Specifying Curves from Points

- Want to modulate curves via “control” points.
- Strategy depends on application. Possibilities:
  - Force a polynomial of degree N-1 through N points (Lagrange interpolate)
  - Specify a combination of “anchor” points and derivatives (Hermite interpolate)
  - Other “blends” (Bézier, B-splines)--more useful than Lagrange/Hermite

Specifying Curves from Points-II

- Issues:
  - Continuity of curve and derivatives (geometric, parametric)
  - Local versus global control
  - Polynomials verses other forms
  - Higher polynomial degree versus stitching lower order polynomials together
  - Polynomial degree (usually 3--fewer is not flexible enough, and higher gives hard to control wiggles).

Parametric vs Geometric Continuity

- **Parametric continuity**:
  - The curve and derivatives up to k are continuous *as a function of parameter value*
  - C^k
  - Useful for (for example) animation
- **Geometric continuity**
  - curve, derivatives up to k'th are the same for equivalent parameter values
  - D^k
  - i.e. there exists a reparametrisation that would achieve parametric continuity
  - Useful, because we often don't require parametric continuity,

Lagrange Interpolate

- Construct a parametric curve that passes through (interpolates) a set of points.
- Lagrange interpolate:
  - give parameter values associated with each point
  - use Lagrange polynomials (one at the relevant point, zero at all others) to construct curve
  - curve is:

$$\sum_{i \in \text{points}} p_i \phi_i^{(l)}(t)$$

- Degree is (#pts-1)
  - e.g. line through two points
  - quadratic through three.
- Functions phi are known as “blending functions”

Hermite curves

- Hermite interpolate
  - Curve passes through specified points **and** has specified derivatives at those points.
  - curve is:
- Use Hermite polynomials to construct curve
  - one at some parameter value and zero at others or
  - derivative one at some parameter value, and zero at others

$$\sum_{i \in \text{points}} p_i \phi_i^{(h)}(t) + \sum_{i \in \text{points}} v_i \phi_i^{(hd)}(t)$$

Blended curves (§9.2)

- Assume degree 3
- Includes Hermite (§9.2.1), Bézier (§9.2.2), and others

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = CT = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Blended curves (§9.2)

- Assume degree 3
- Includes Hermite (§9.2.1), Bézier (§9.2.2), and others

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = [G_1 \quad G_2 \quad G_3 \quad G_4] \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Hermite (§9.2.2)

- Geometry matrix
  - First two columns are endpoints
  - Next two columns are derivatives at those points

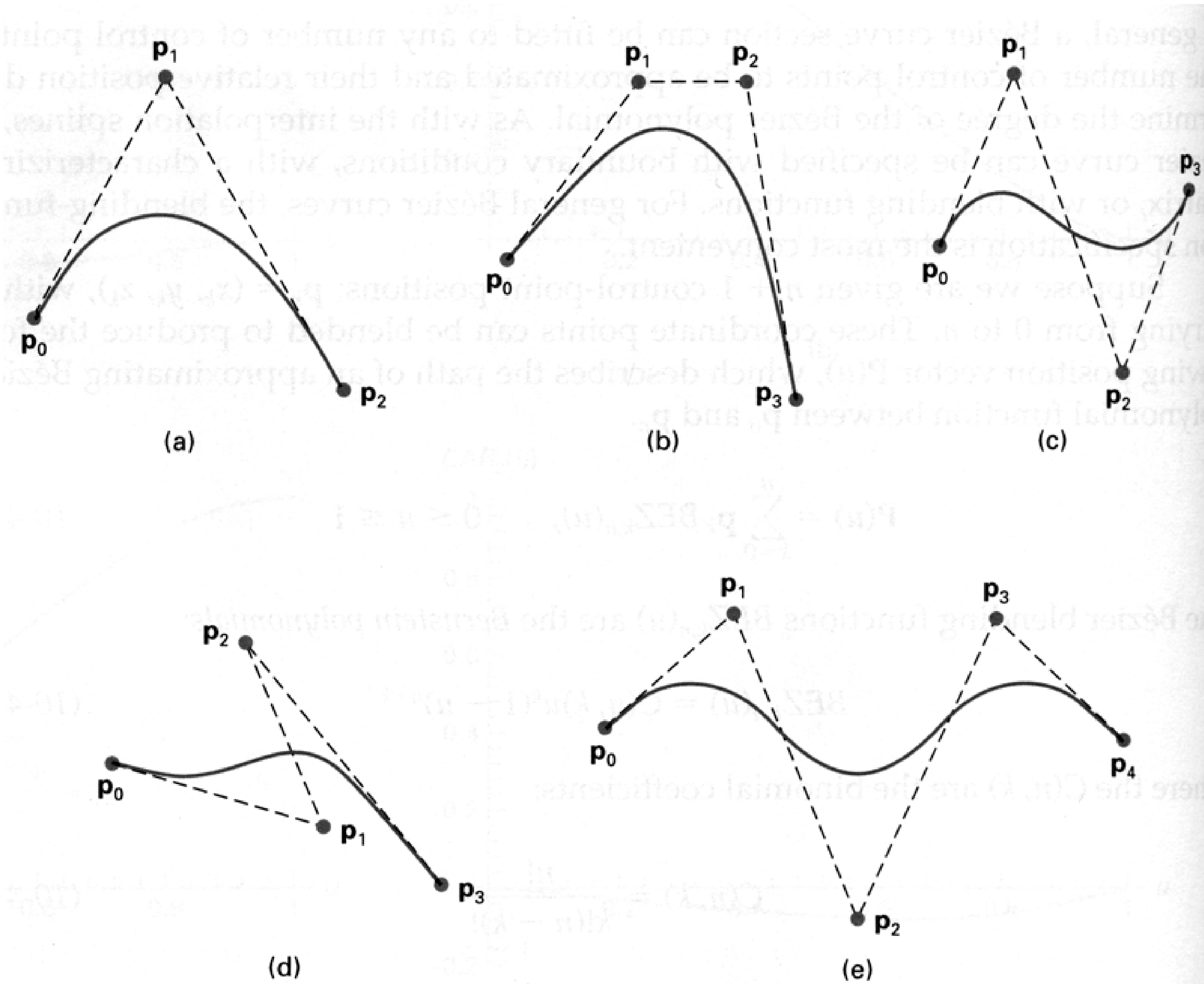
$$M_H = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Derivation?
See page 333

Bézier (§9.2.3)

- Curve goes through two control points
- Curve is adjusted by moving two (cubic case) other control points
- Tangent at endpoints is in direction of adjacent control point
- Curve lies in convex hull of all 4 (cubic case) control points.
  - First two columns are endpoints
  - Next two columns are derivatives at those points

Example Bézier Curves



Bézier (§9.2.2)

- Geometry matrix
  - First two columns are endpoints
  - Next two are like derivatives from the Hermite case, but are now defined by

$$R_1 = 3(P_2 - P_1)$$
$$R_2 = 3(P_4 - P_3)$$

  - Note that this gives our condition on endpoint tangents
  - Factor of 3 gives good “balance” in control point effect (see book exercise 9.9), and is needed to be consistent with other derivations (e.g., Bernstein polynomials, subdivision, etc).

Bézier in standard form (summary)

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} P_1 & P_4 & R_1 & R_2 \end{bmatrix} M_B \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

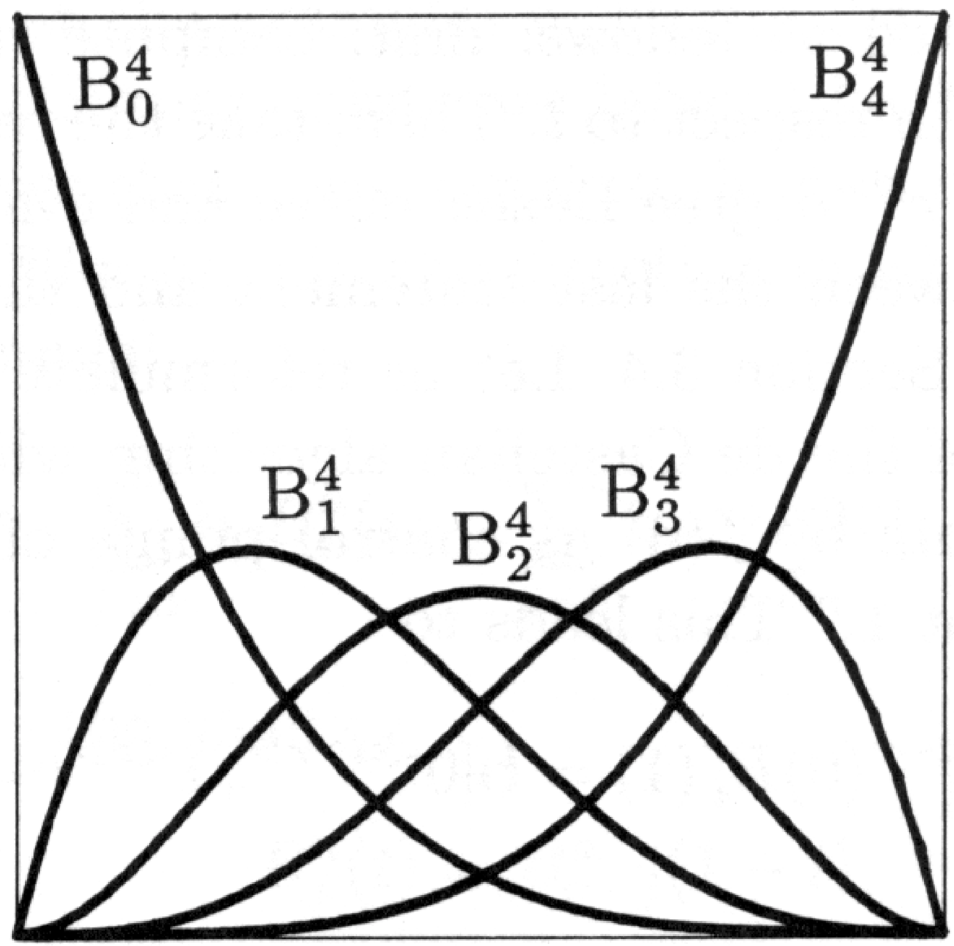
$$M_B = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bézier curves - II

- Blending functions are the Bernstein polynomials

$$c(t) = \sum_{i=0}^{n} p_i B_i^n(t)$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

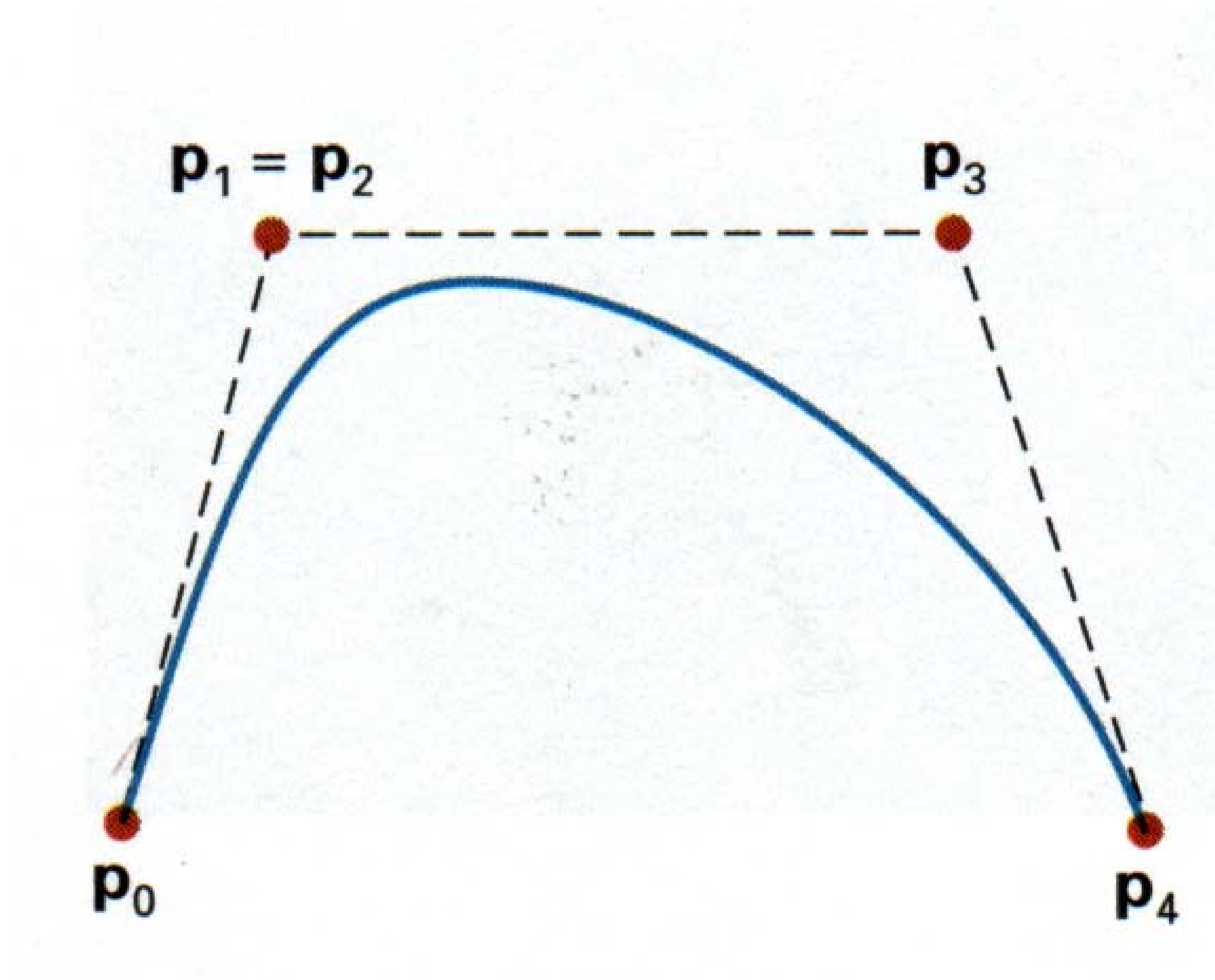


Bézier curves - III

- Bernstein polynomials have several important properties
  - they sum to 1, hence curve lies within convex hull of control points
  - curve interpolates its endpoints
  - curve’s tangent at start lies along the vector from p0 to p1
  - tangent at end lies along vector from pn-1 to pn

Bézier curve tricks - I

- "Pull" a curve toward a control point by doubling the control point



Bézier curve tricks-II

- Close the curve by making last point and first point coincident
  - curve has continuous tangent if first segment and last segment are collinear