Blended curves (§9.2)

- Assume degree 3
- Includes Hermite (§9.2.1), Bézier (§9.2.2), and others

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = CT = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Blended curves (§9.2)

- Assume degree 3
- Includes Hermite (§9.2.1), Bézier (§9.2.2), and others

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = [G_1 \quad G_2 \quad G_3 \quad G_4] \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Hermite (§9.2.2)

- Geometry matrix
  - First two columns are endpoints
  - Next two columns are derivatives at those points

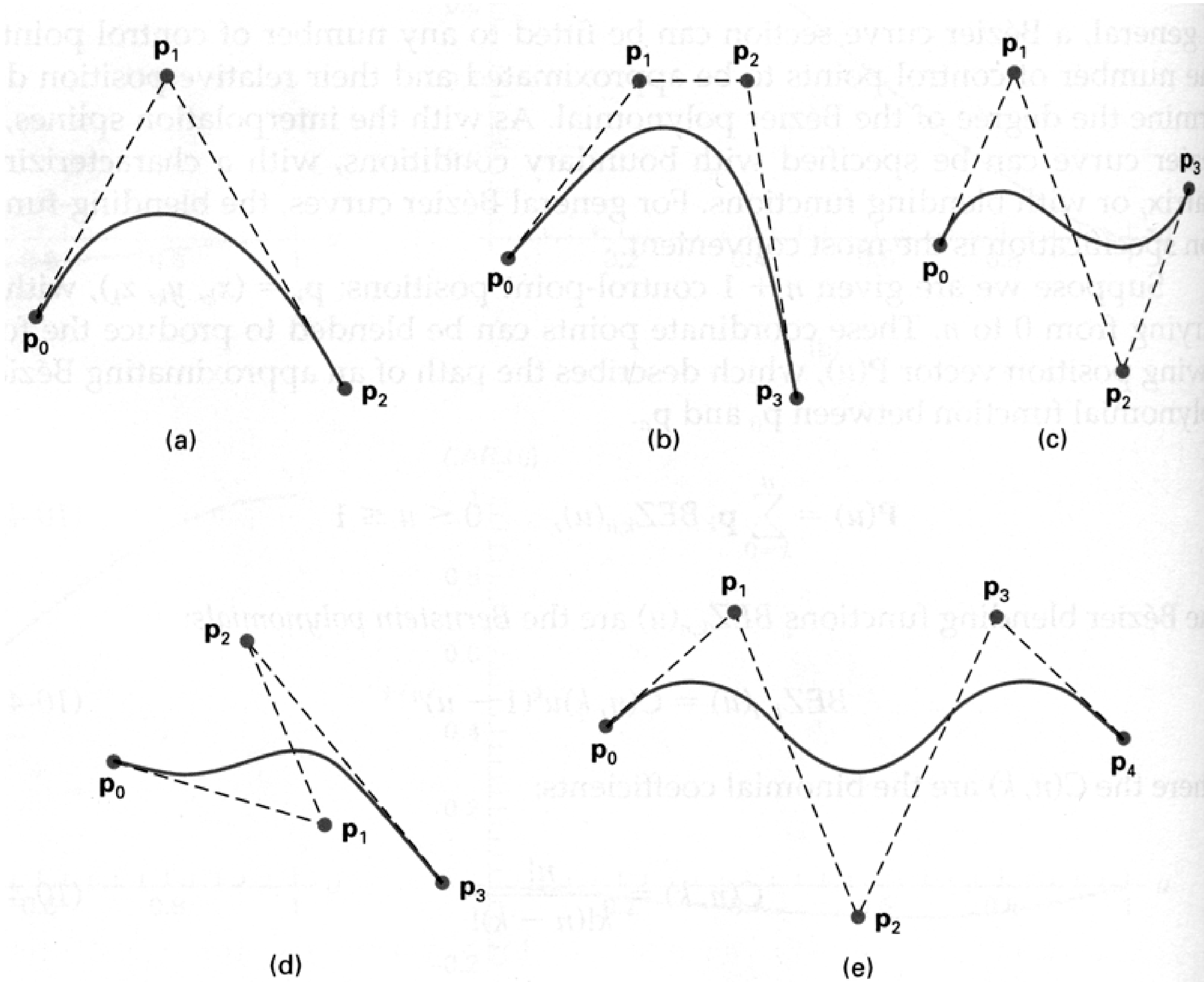
$$M_H = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Derivation?
See page 333

Bézier (§9.2.3)

- Curve goes through two control points
- Curve is adjusted by moving two (cubic case) other control points
- Tangent at endpoints is in direction of adjacent control point
- Curve lies in convex hull of all 4 (cubic case) control points.
  - First two columns are endpoints
  - Next two columns are derivatives at those points

Example Bézier Curves



Bézier (§9.2.2)

- Geometry matrix
  - First two columns are endpoints
  - Next two are like derivatives from the Hermite case, but are now defined by

$$R_1 = 3(P_2 - P_1)$$
$$R_2 = 3(P_4 - P_3)$$

  - Note that this gives our condition on endpoint tangents
  - Factor of 3 gives good "balance" in control point effect (see book exercise 9.9), and is needed to be consistent with other derivations (e.g., Bernstein polynomials, subdivision, etc).

$$R_1 = 3(P_2 - P_1)$$
$$R_2 = 3(P_4 - P_3)$$

Means that

$$\begin{bmatrix} P_1 & P_4 & R_1 & R_2 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

or

$$\begin{bmatrix} P_1 & P_4 & R_1 & R_2 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} M_{HB}$$

where

$$M_{HB} = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Recall Hermite

$$Q(t)=\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}=\begin{bmatrix} P_1 & P_4 & R_1 & R_2 \end{bmatrix} M_H \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

From previous slide

$$\begin{bmatrix} P_1 & P_4 & R_1 & R_2 \end{bmatrix}=\begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} M_{HB}$$

So, for Bézier

$$Q(t)=\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}=\begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} M_{HB} M_H \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Want M_B in $Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} M_B \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$

$$M_B = M_{HB} M_B$$

$$= \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bézier in standard form (summary)

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} M_B \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

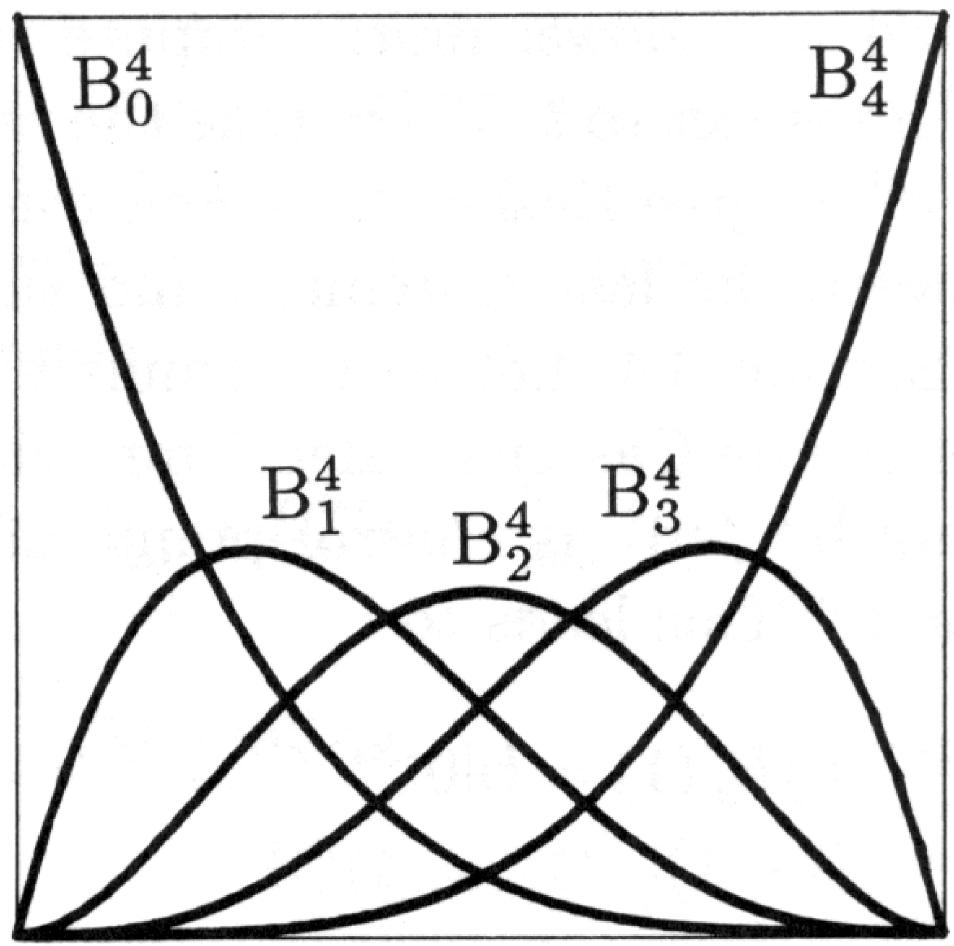
$$M_B = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bézier curves - II

- Blending functions are the Bernstein polynomials

$$c(t) = \sum_{i=0}^{n} p_i B_i^n(t)$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

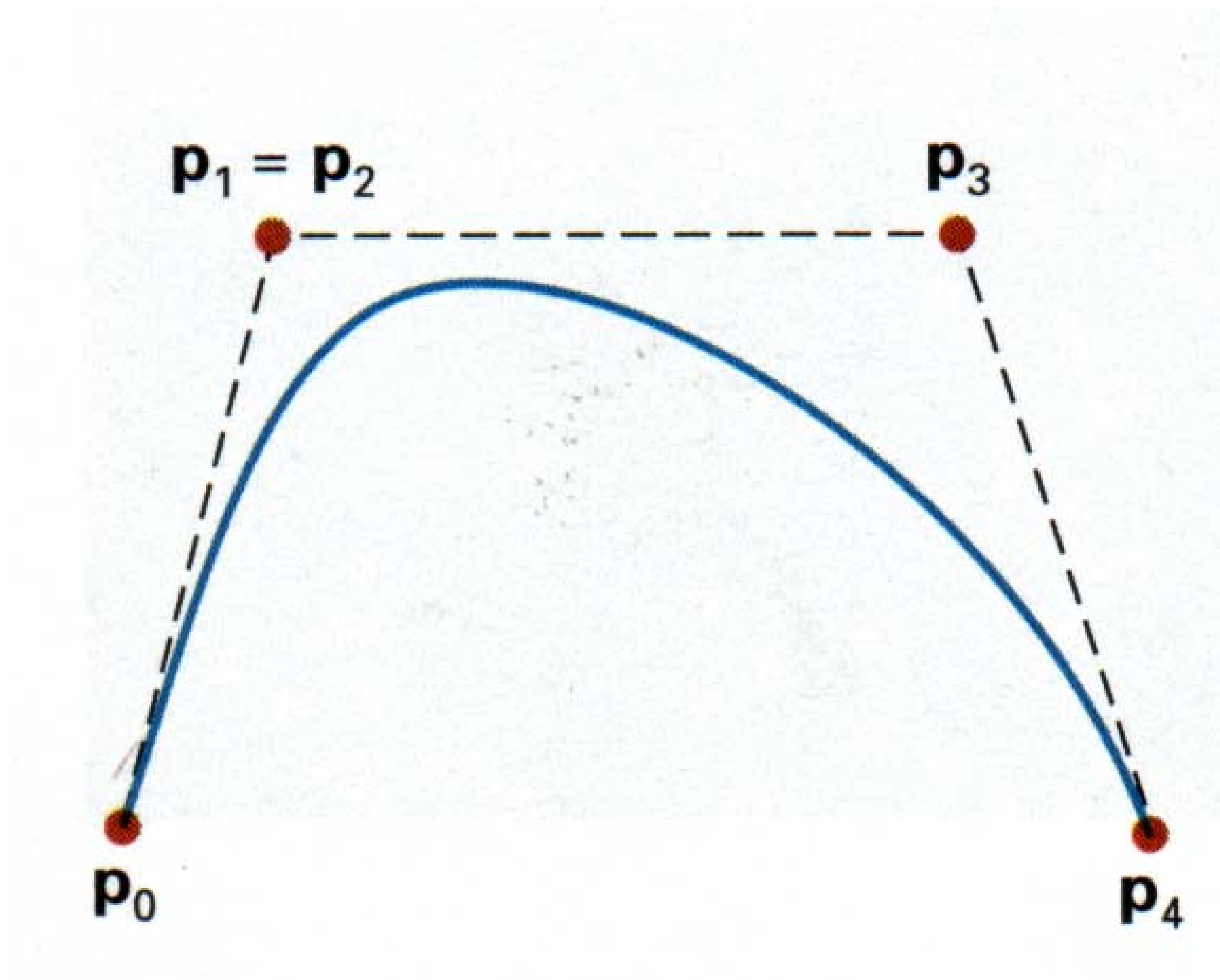


Bézier curves - III

- Bernstein polynomials have several important properties
  - they sum to 1, hence curve lies within convex hull of control points
  - curve interpolates its endpoints
  - curve's tangent at start lies along the vector from p0 to p1
  - tangent at end lies along vector from pn-1 to pn

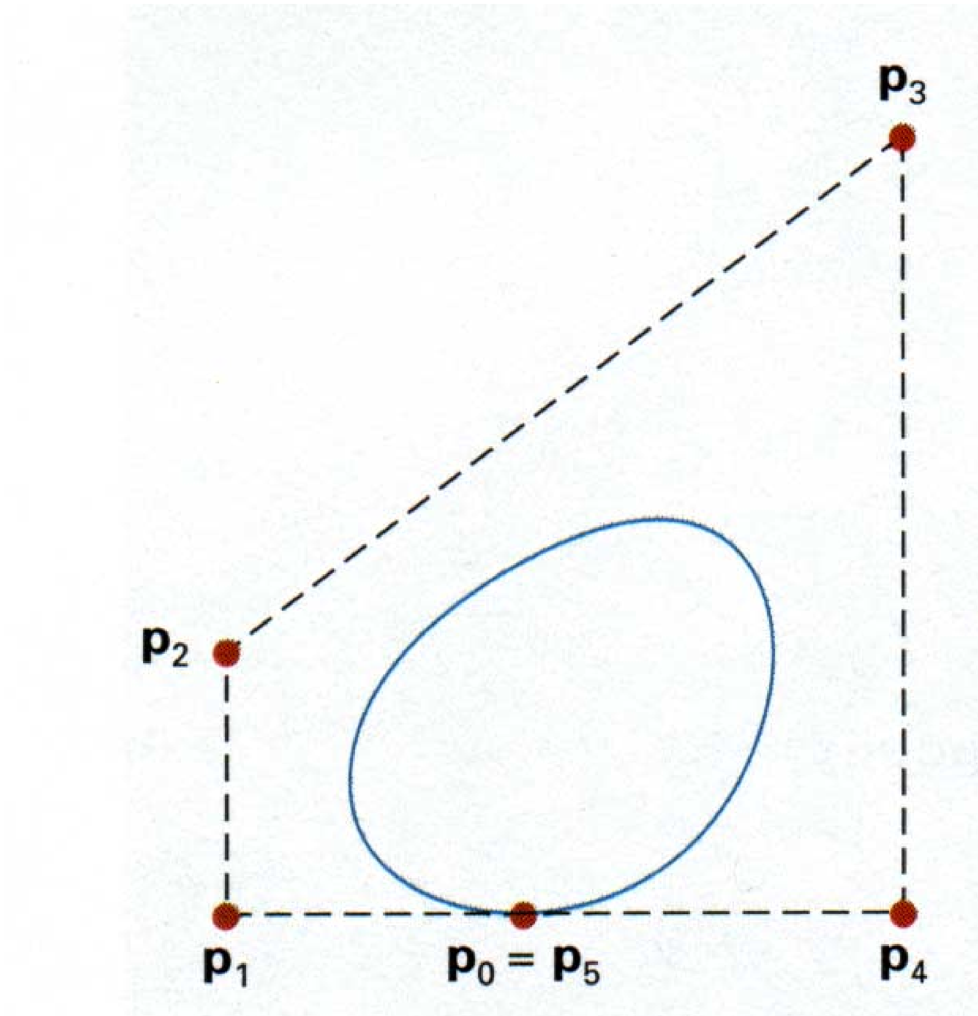
Bézier curve tricks - I

- "Pull" a curve toward a control point by doubling the control point



Bézier curve tricks-II

- Close the curve by making last point and first point coincident
  - curve has continuous tangent if first segment and last segment are collinear



Interpolating Splines

- Key idea:
  - high degree interpolates are badly behaved->
  - construct curves out of low degree segments

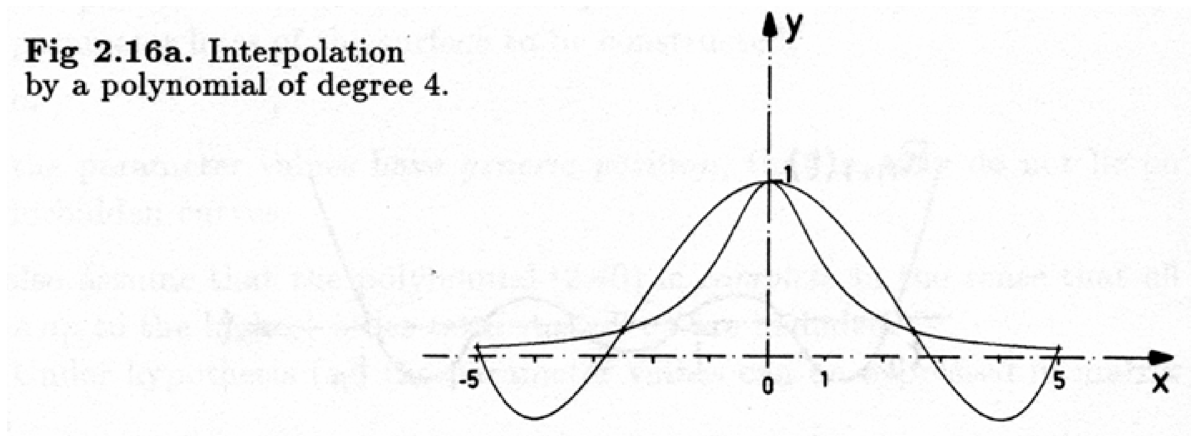


Fig 2.16a. Interpolation by a polynomial of degree 4.

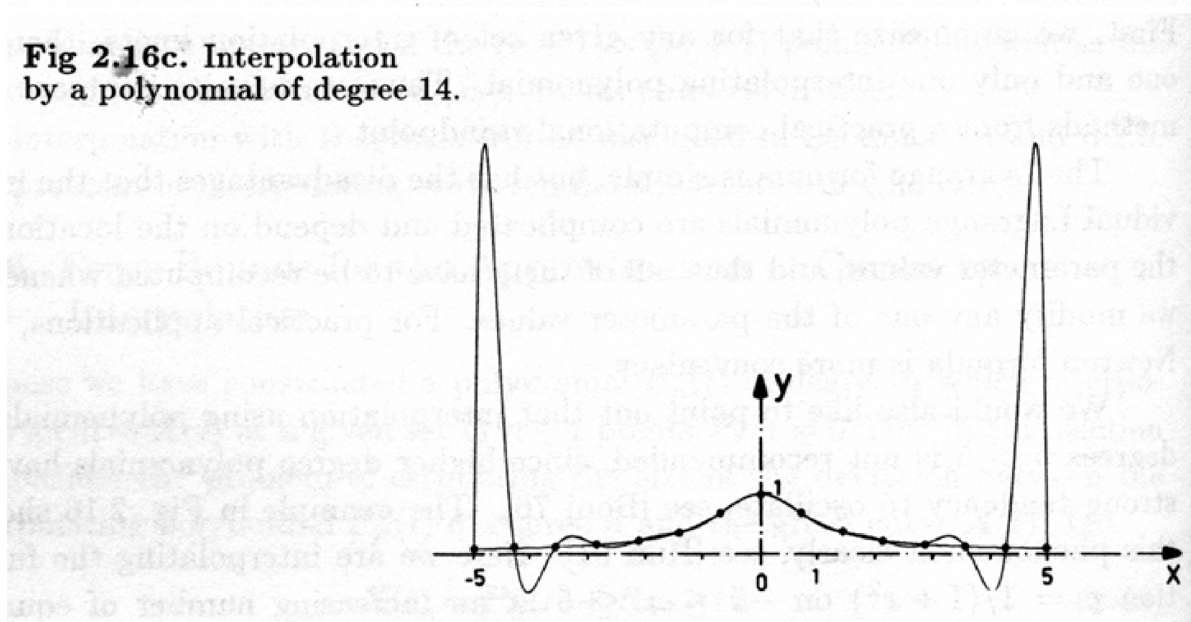


Fig 2.16c. Interpolation by a polynomial of degree 14.

Interpolating Splines - II

- n+1 points;
- write derivatives X'
- X_i is spline for interval between P_i and P_{i+1}

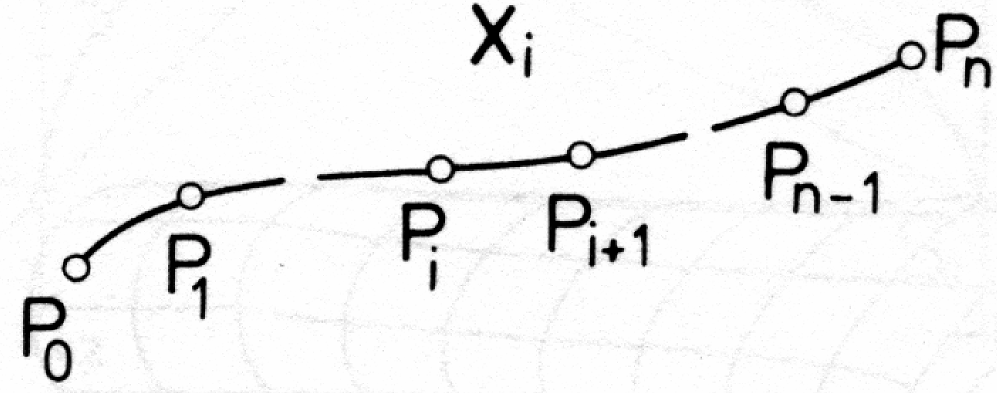


Fig. 3.11. The spline segment $\boldsymbol{X}_i$.

Interpolating Splines - II

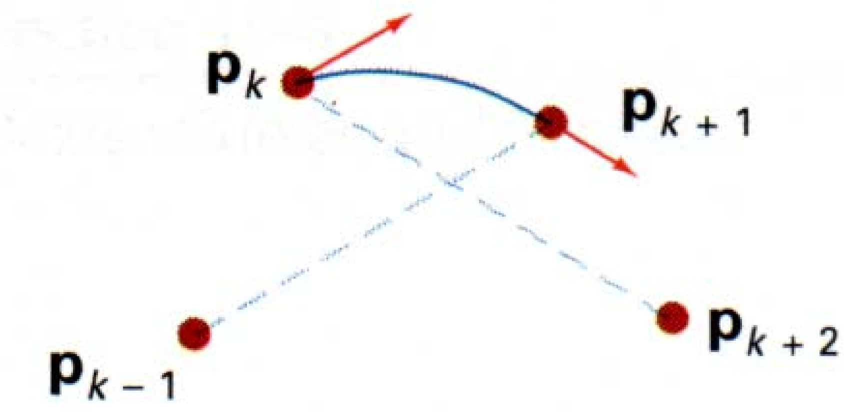
- Bolt together a series of Hermite curves with derivatives matching at joints (Knots).
- But where are the derivative values to come from?
  - Measurements
  - Combination of points (see cardinal splines--next topic)
  - Continuity considerations
  - Conventions for enpoints

- Cardinal splines

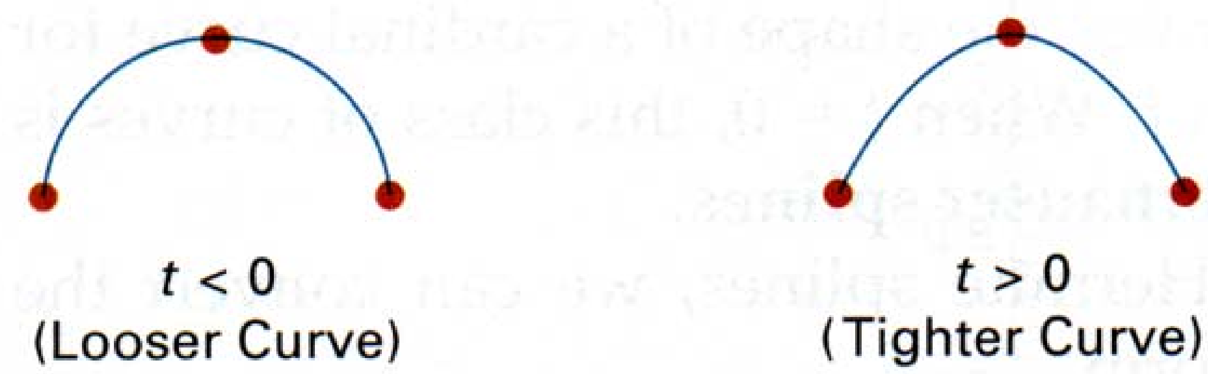
$$P'_k = \left(\frac{1}{2}\right)(1-t)\left(P_{k+1} - P_{k-1}\right)$$

  - t is “tension”
  - still need to specify endpoint tangents
    - or use difference between first two, last two points

Tension



- larger values of tension give tighter curves (limit is linear interpolate).



Interpolating Splines

- Intervals:

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b.$$

$$\Delta t_i := t_{i+1} - t_i.$$

  - t values often called “knots”

- Spline form:

$$\boldsymbol{X}_i(t) := \boldsymbol{A}_i(t-t_i)^3 + \boldsymbol{B}_i(t-t_i)^2 + \boldsymbol{C}_i(t-t_i) + \boldsymbol{D}_i,$$
$$t \in [t_i, t_{i+1}], \qquad i = 0(1)N\text{–}1,$$

Continuity

- Require at endpoints:
  - endpoints equal
  - 1'st derivatives equal
  - 2'nd derivatives equal
- Now we get extra information from continuity (instead of tension equation, tangent measurements, etc)

$$
\begin{aligned}
\boldsymbol{X}_i(t_i) &= \boldsymbol{X}_{i-1}(t_i) \quad \text{or} \quad \boldsymbol{X}_i(t_{i+1}) = \boldsymbol{X}_{i+1}(t_{i+1}),\\
\boldsymbol{X}'_i(t_i) &= \boldsymbol{X}'_{i-1}(t_i) \quad \text{or} \quad \boldsymbol{X}'_i(t_{i+1}) = \boldsymbol{X}'_{i+1}(t_{i+1}),\\
\boldsymbol{X}''_i(t_i) &= \boldsymbol{X}''_{i-1}(t_i) \quad \text{or} \quad \boldsymbol{X}''_i(t_{i+1}) = \boldsymbol{X}''_{i+1}(t_{i+1}).
\end{aligned}
$$

- From endpoint and 1'st derivative:

$$\boldsymbol{X}_i(t_i) = \boldsymbol{P}_i = \boldsymbol{D}_i, \qquad \boldsymbol{X}_i(t_{i+1}) = \boldsymbol{P}_{i+1} = \boldsymbol{A}_i \Delta t_i^3 + \boldsymbol{B}_i \Delta t_i^2 + \boldsymbol{C}_i \Delta t_i + \boldsymbol{D}_i,$$
$$\boldsymbol{X}'_i(t_i) = \boldsymbol{P}'_i = \boldsymbol{C}_i, \qquad \boldsymbol{X}'_i(t_{i+1}) = \boldsymbol{P}'_{i+1} = 3\boldsymbol{A}_i \Delta t_i^2 + 2\boldsymbol{B}_i \Delta t_i + \boldsymbol{C}_i,$$

- So that

$$\boldsymbol{A}_i = \frac{1}{(\Delta t_i)^3}[2(\boldsymbol{P}_i - \boldsymbol{P}_{i+1}) + \Delta t_i(\boldsymbol{P}'_i + \boldsymbol{P}'_{i+1})],$$
$$\boldsymbol{B}_i = \frac{1}{(\Delta t_i)^2}[3(\boldsymbol{P}_{i+1} - \boldsymbol{P}_i) - \Delta t_i(2\boldsymbol{P}'_i + \boldsymbol{P}'_{i+1})].$$

- Yielding:

$$\boldsymbol{X}_i(t) =$$
$$\boldsymbol{P}_i\left(2\frac{(t-t_i)^3}{(\Delta t_i)^3} - 3\frac{(t-t_i)^2}{(\Delta t_i)^2} + 1\right) + \boldsymbol{P}_{i+1}\left(-2\frac{(t-t_i)^3}{(\Delta t_i)^3} + 3\frac{(t-t_i)^2}{(\Delta t_i)^2}\right)$$
$$+ \boldsymbol{P}'_i\left(\frac{(t-t_i)^3}{(\Delta t_i)^2} - 2\frac{(t-t_i)^2}{\Delta t_i} + (t-t_i)\right) + \boldsymbol{P}'_{i+1}\left(\frac{(t-t_i)^3}{(\Delta t_i)^2} - \frac{(t-t_i)^2}{\Delta t_i}\right)$$

- Second Derivative:

$$\boldsymbol{X}_i''(t) = 6\boldsymbol{P}_i\left(\frac{2(t-t_i)}{(\Delta t_i)^3} - \frac{1}{(\Delta t_i)^2}\right) + 6\boldsymbol{P}_{i+1}\left(-2\frac{(t-t_i)}{(\Delta t_i)^3} + \frac{1}{(\Delta t_i)^2}\right) + 2\boldsymbol{P}_i'\left(3\frac{(t-t_i)}{(\Delta t_i)^2} - \frac{2}{\Delta t_i}\right) + 2\boldsymbol{P}_{i+1}'\left(\frac{3(t-t_i)}{(\Delta t_i)^2} - \frac{1}{\Delta t_i}\right).$$

- Want: $\boldsymbol{X}_{i-1}''(t_i) = \boldsymbol{X}_i''(t_i)$

$$\Delta t_i \boldsymbol{P}_{i-1}' + 2(\Delta t_{i-1} + \Delta t_i)\boldsymbol{P}_i' + \Delta t_{i-1}\boldsymbol{P}_{i+1}' = 3\frac{\Delta t_{i-1}}{\Delta t_i}(\boldsymbol{P}_{i+1} - \boldsymbol{P}_i) + 3\frac{\Delta t_i}{\Delta t_{i-1}}(\boldsymbol{P}_i - \boldsymbol{P}_{i-1}).$$

Missing equations

- Recurrence relations represent d(n-1) equations in d(n+1) unknowns (d is dimension)
- We need to supply the derivative at the start and at the finish (or two equivalent constraints)
- Options:
  - second derivatives vanish at each end (natural spline)
  - give slopes at the boundary
    - vector from first to second, second last to last
    - parabola through first three, last three points
  - third derivative is the same at first, last knot

B-splines - I

- Now consider stitching together curves which do not necessarily pass through the control points (i.e., back to blending functions).
- Local control
- Blending functions are non-zero over limited range--thus they are like “switches”
- In the simplest case of uniformly spaced control points, the blending functions will be shifted versions of the same function.

B-splines - II

- Curve (general case):

$$X(t) = \sum_{k=0}^{n} P_k B_{k,d}(t)$$

- The "order" d is:

$$2 \leq d \leq n + 1$$

- Usual case: n is 4, d is 3.

B-Spline Blending Functions (§9.2.5)

- Knots
  - parameter values where curve segments meet, as in Hermite example

$$\left(t_0, t_1, \ldots, t_{n+d}\right)$$

where $t_0 \le t_1 \le \ldots \le t_{n+d}$

- Blending functions

$$B_{k,1}(t) = \begin{cases} 1 & t_k \le t \le t_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k}\right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}}\right) B_{k+1,d-1}(t)$$

- Division by 0 gives 0

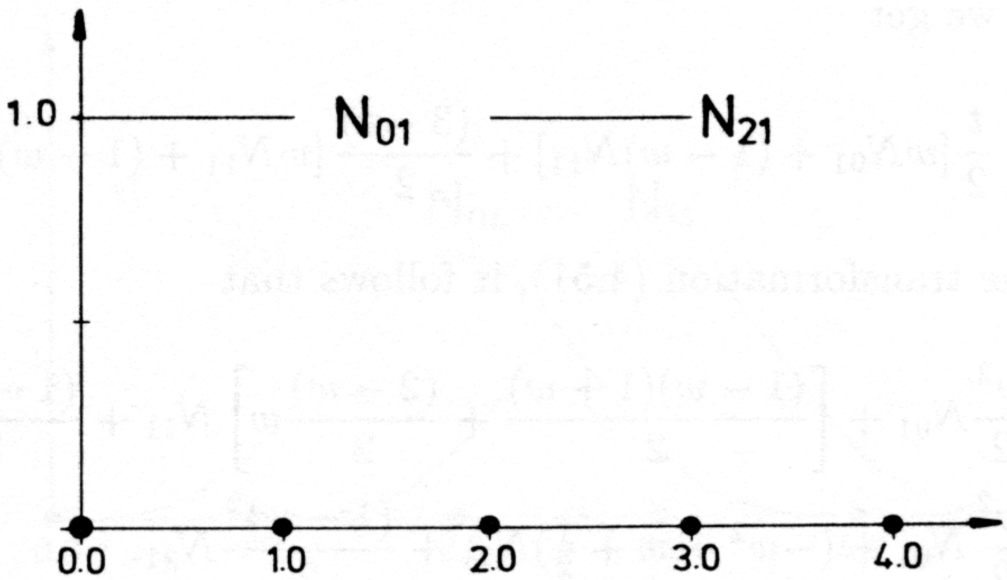


Fig. 4.22c. The B-splines N_{01}, N_{21}.

These figures show blending functions with a uniform knot vector, knots at 0, 1, 2, etc. Note that N is the same as our B

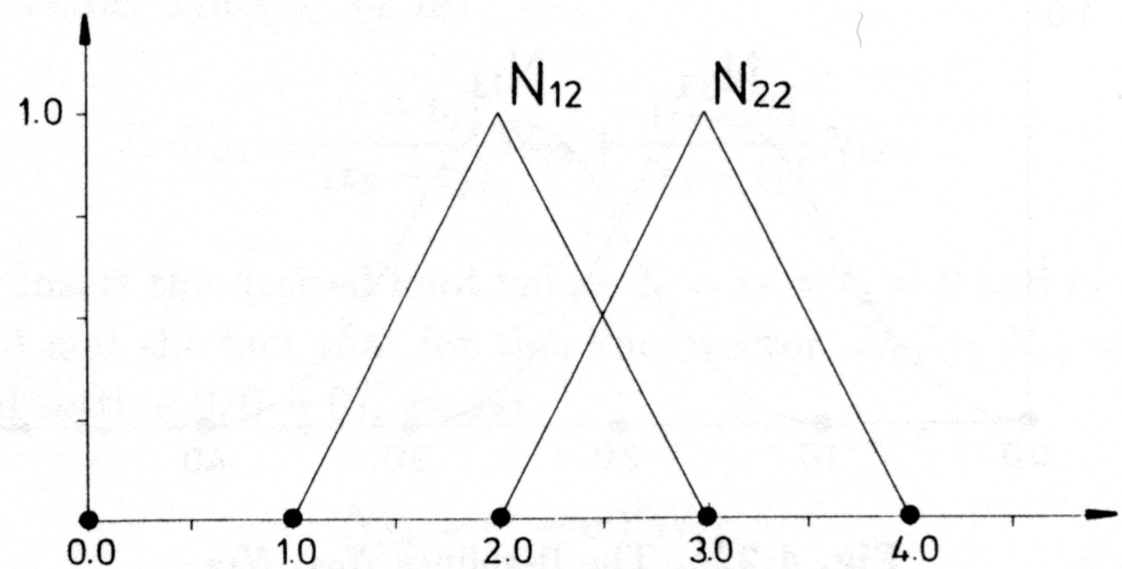


Fig. 4.22d. The B-splines N_{12}, N_{22}.