B-splines - I

- Now consider stitching together curves which do not necessarily pass through the control points (i.e., back to blending functions).
- Local control
- Blending functions are non-zero over limited range--thus they are like “switches”
- In the simplest case of uniformly spaced control points, the blending functions will be shifted versions of the same function.

B-splines - II

- Curve (general case):

$$X(t) = \sum_{k=0}^{n} P_k B_{k,d}(t)$$

- The “order” d is:

$$2 \le d \le n + 1$$

- Usual case: n is 4, d is 3.

B-Spline Blending Functions (§9.2.5)

- Knots
  - parameter values where curve segments meet, as in Hermite example

$$\left(t_0, t_1, \ldots, t_{n+d}\right)$$

where $t_0 \le t_1 \le \ldots \le t_{n+d}$

- Blending functions

$$B_{k,1}(t) = \begin{cases} 1 & t_k \le t \le t_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k}\right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}}\right) B_{k+1,d-1}(t)$$

- Division by 0 gives 0

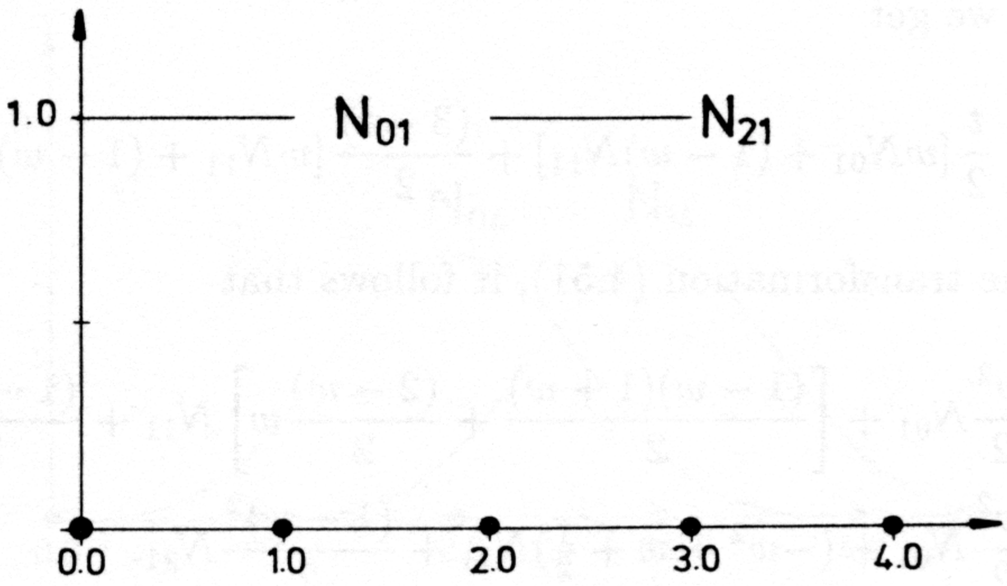


Fig. 4.22c. The B-splines N_{01}, N_{21}.

These figures show blending functions with a uniform knot vector, knots at 0, 1, 2, etc. Note that N is the same as our B

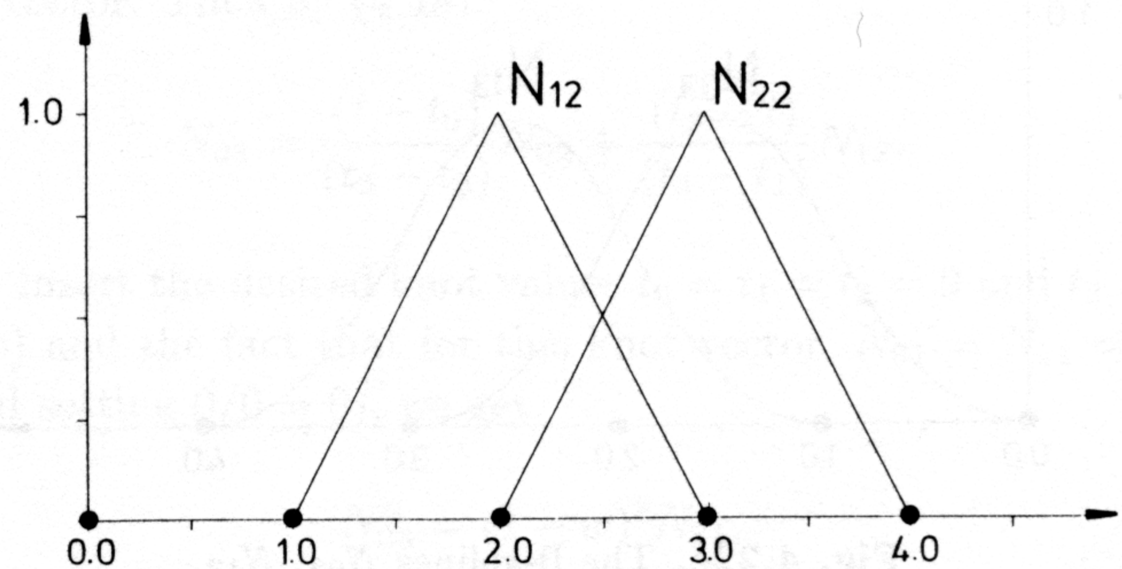


Fig. 4.22d. The B-splines N_{12}, N_{22}.

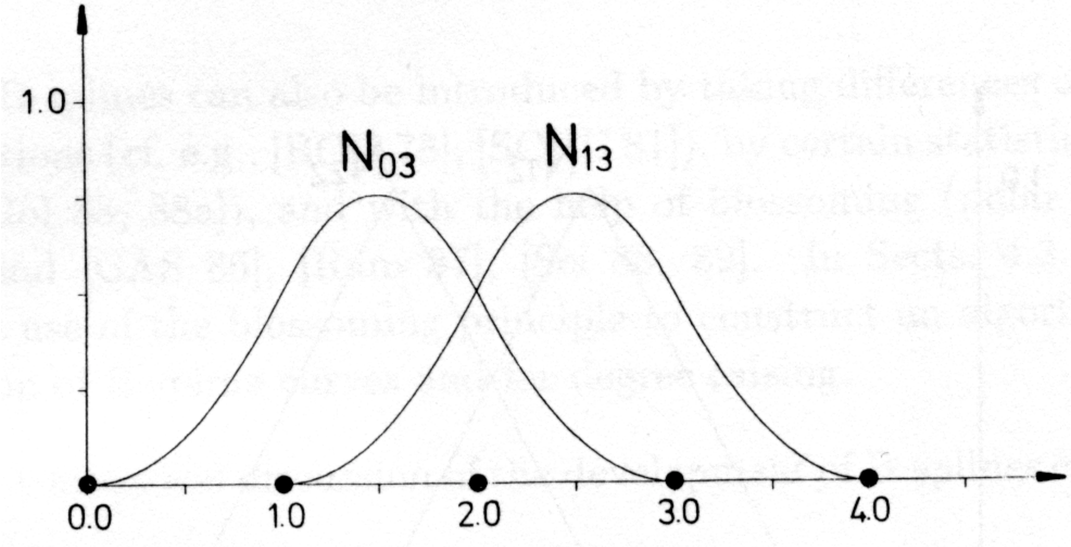
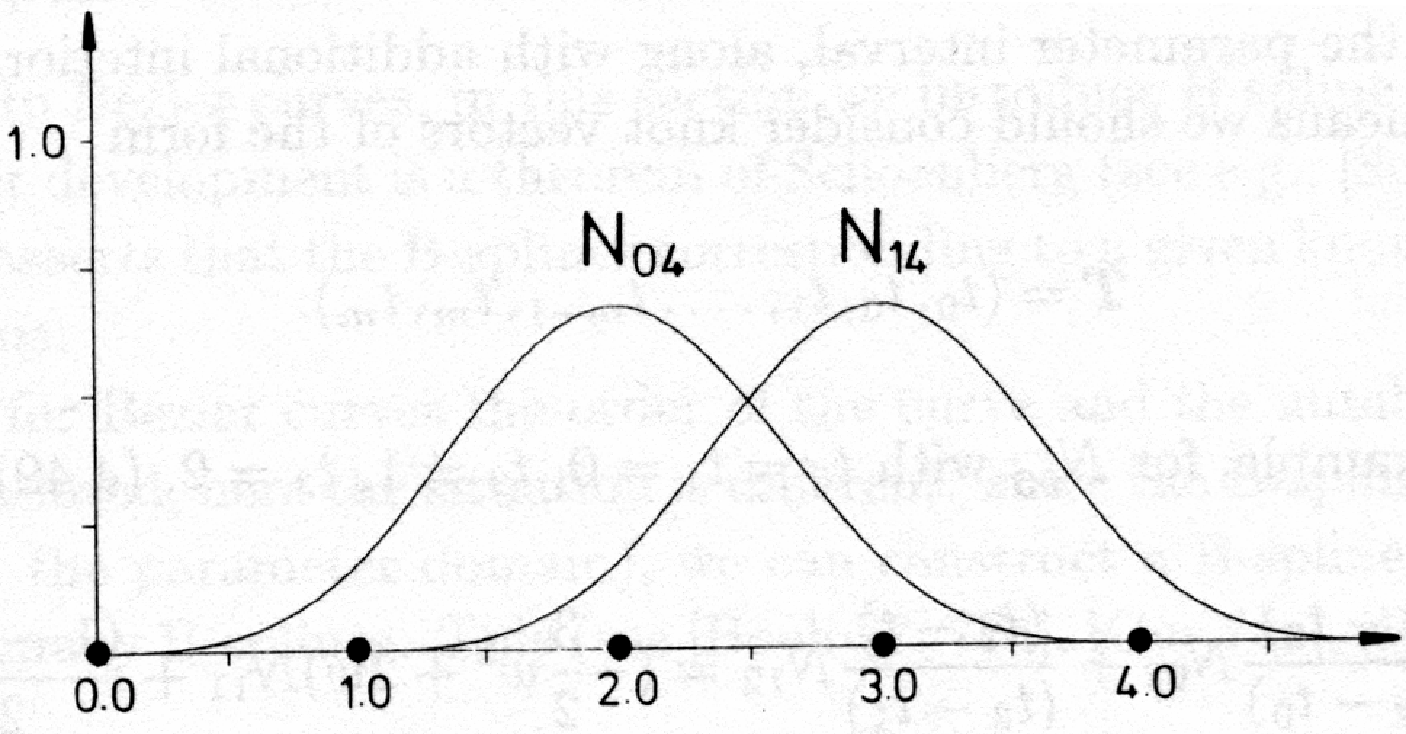


Fig. 4.22e. The B-splines N_{03}, N_{13}.



Matrix form of Uniform Cubic B-Spline Blending Functions

$$M_B = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

Closed B-Splines

- Periodically extend the control points and the knots

$$P_{n+1} = P_0 \qquad t_{n+1} = t_0$$
$$P_{n+2} = P_1 \qquad t_{n+2} = t_1$$
$$....$$
$$P_{n+d-1} = P_{d-2} \qquad t_{n+d-1} = t_{n+d-2}$$

Fig. 4.26a.

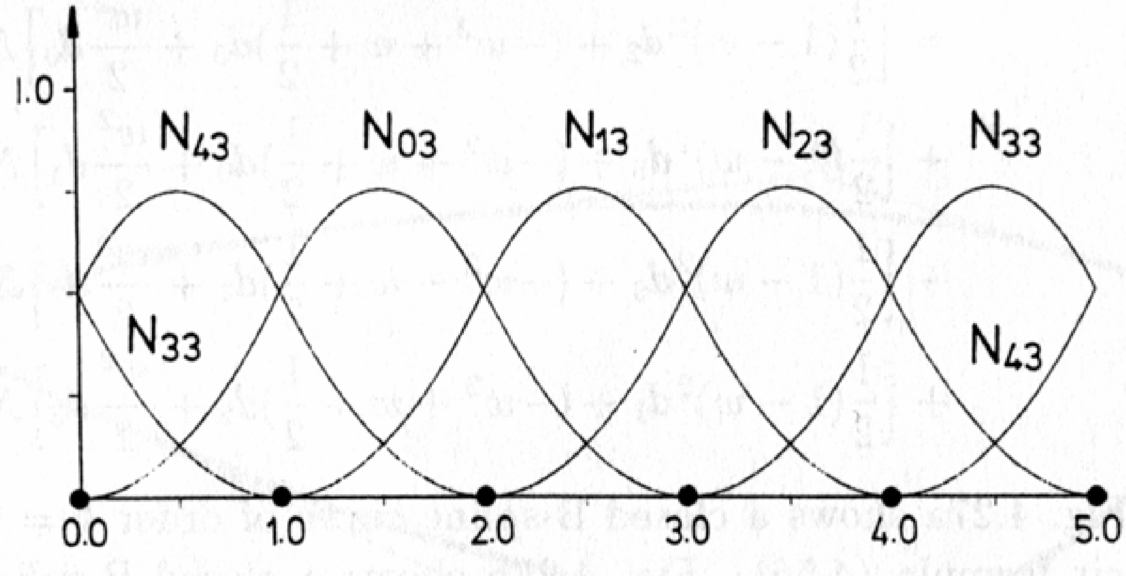


Fig. 4.26b.

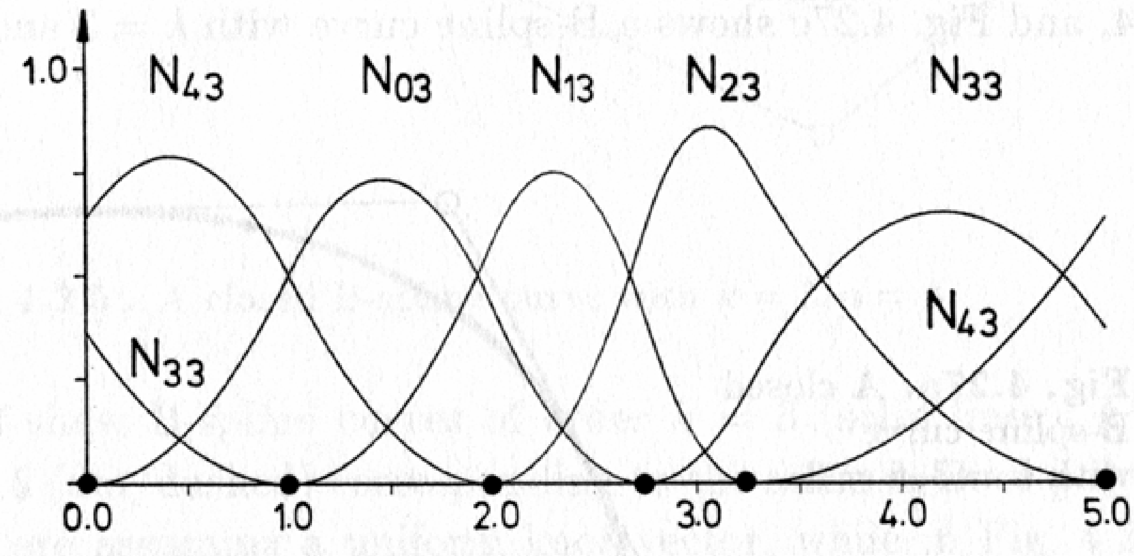


Fig. 4.26. B-splines with uniform and non-uniform knot vectors for a closed B-spline curve.

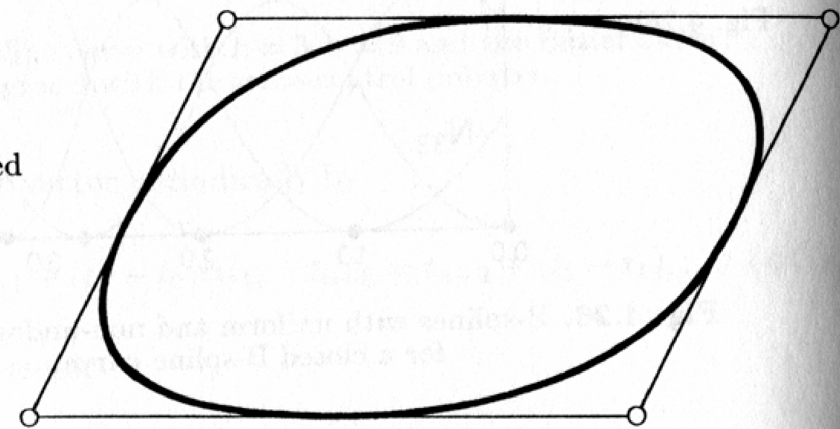


Fig. 4.27a. A closed B-spline curve with $k = 3, n = 3$.

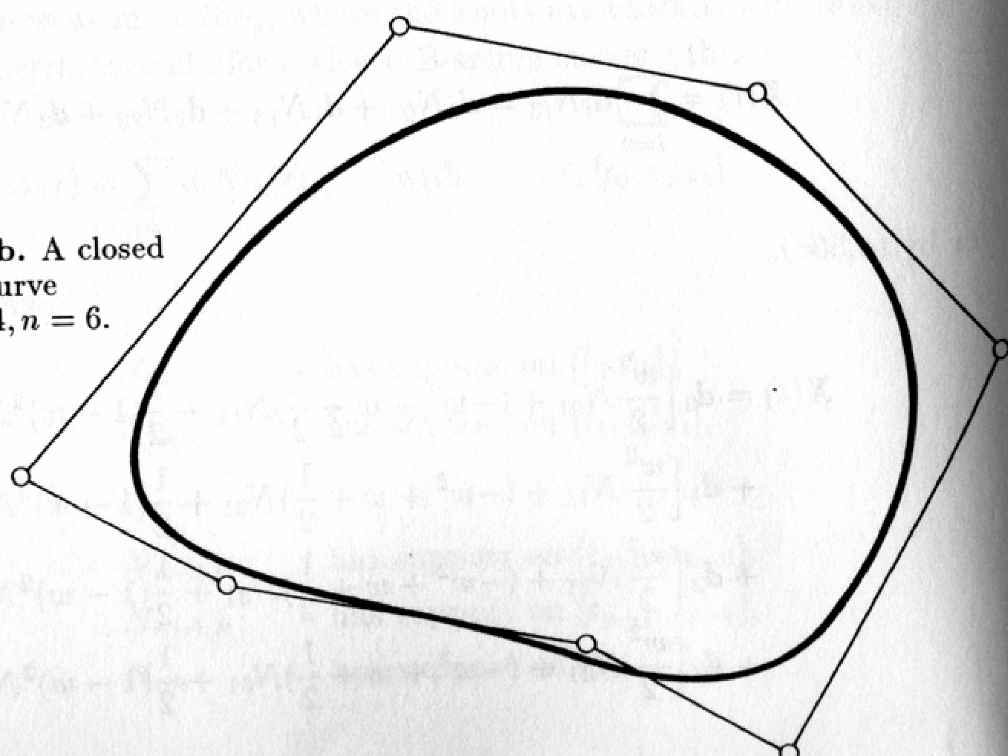


Fig. 4.27b. A closed B-spline curve with $k = 4, n = 6$.

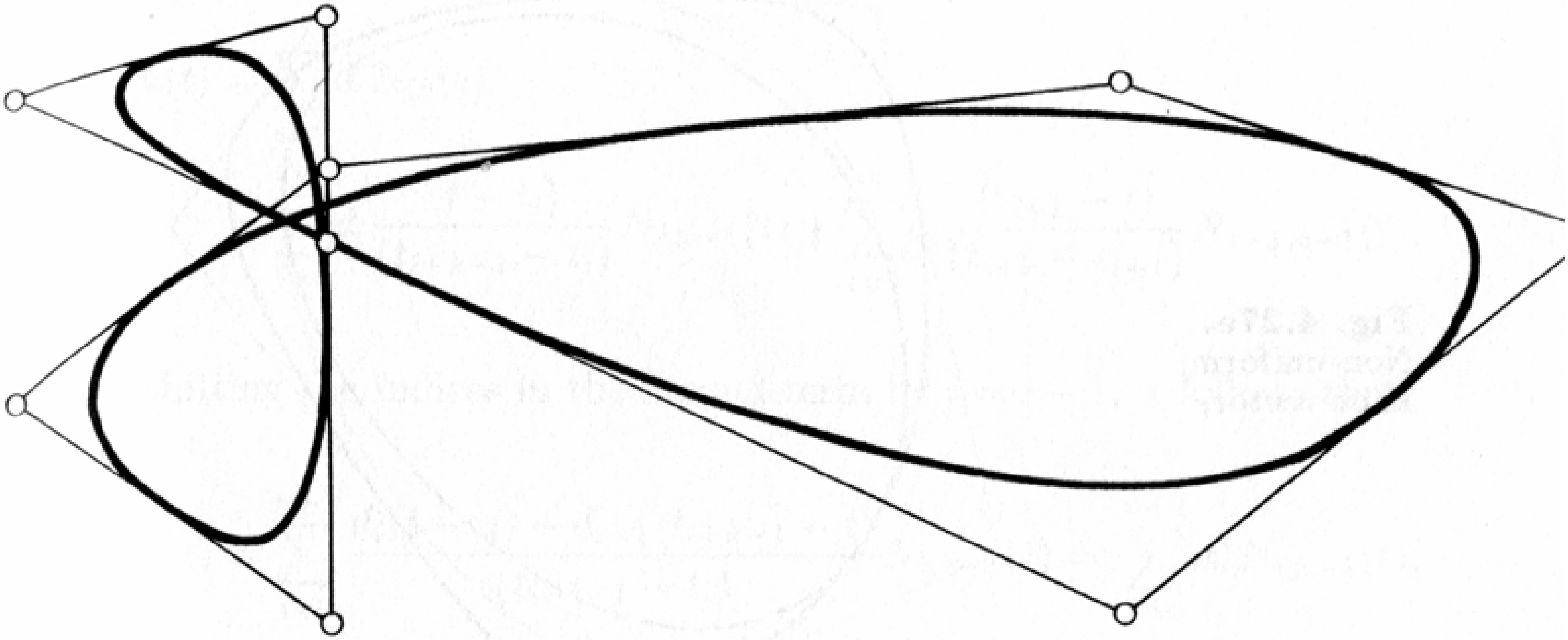


Fig. 4.27c. A closed B-spline curve with $k = 3, n = 8$.

Repeated knots

- Definition works for repeated knots (if we are understanding about 0/0)
- Repeated knot reduces continuity. A B-spline blending function has continuity C^{d-2}; if the knot is repeated m times, continuity is now C^{d-m-1}
- e.g. -> quadratic B-spline (i.e. order 3) with a double knot

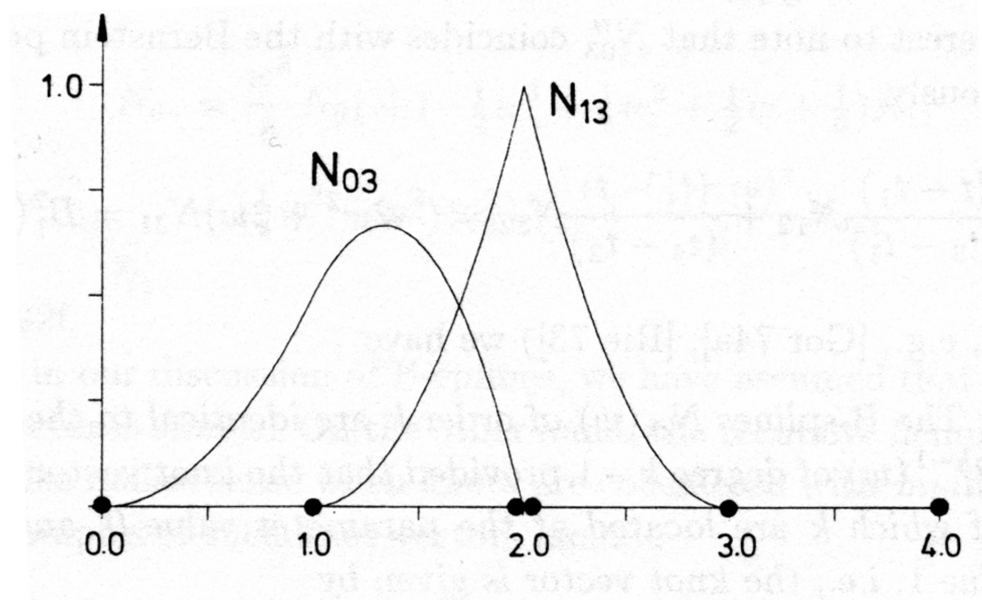


Fig. 4.22g. A quadratic B-spline with a double knot.

Most useful case

- Select the first d and the last d knots to be the same (i.e. repeat)
  - we then get the first and last points lying on the curve
  - also, the curve is tangent to the first and last segment

- e.g. cubic case below
- **Remember** that a control point influences at most d parameter intervals - **local control**

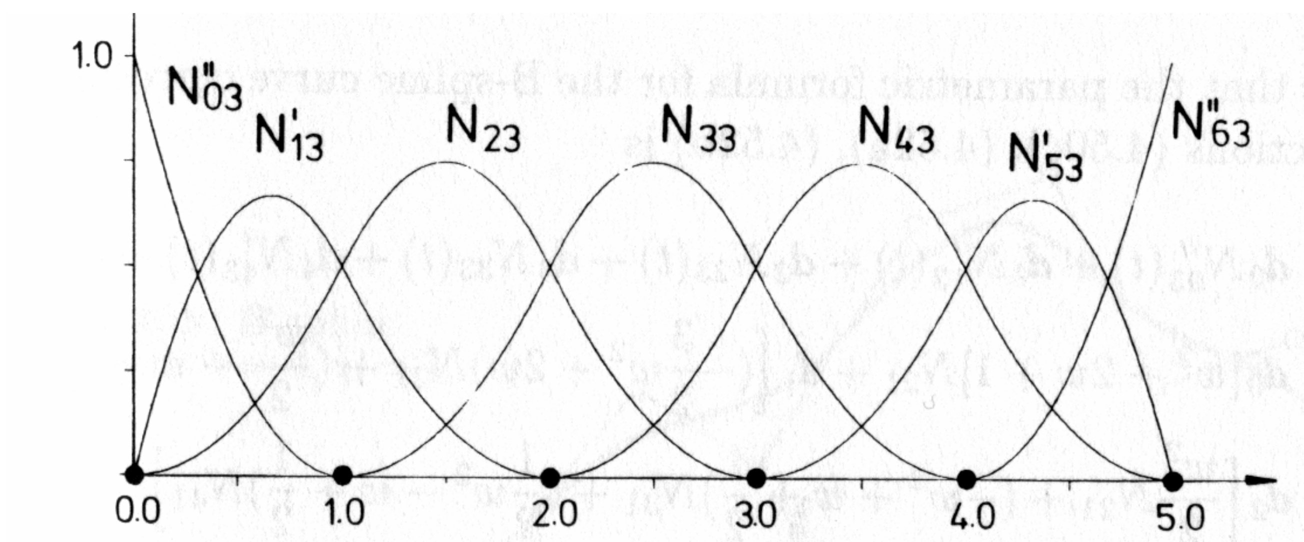


Fig. 4.24a. B-splines for an open B-spline curve with uniform knot vector.

Fig. 4.25a. B-spline curve with $k = 3$, $n = 5$.

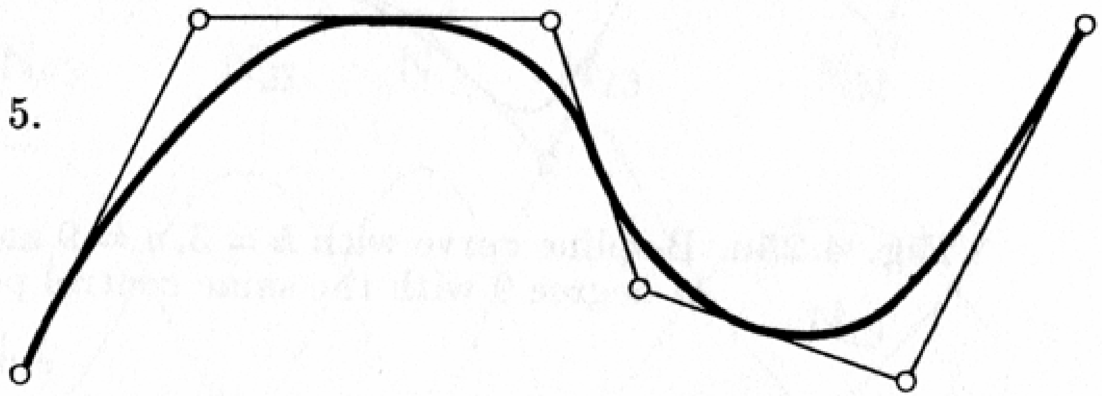
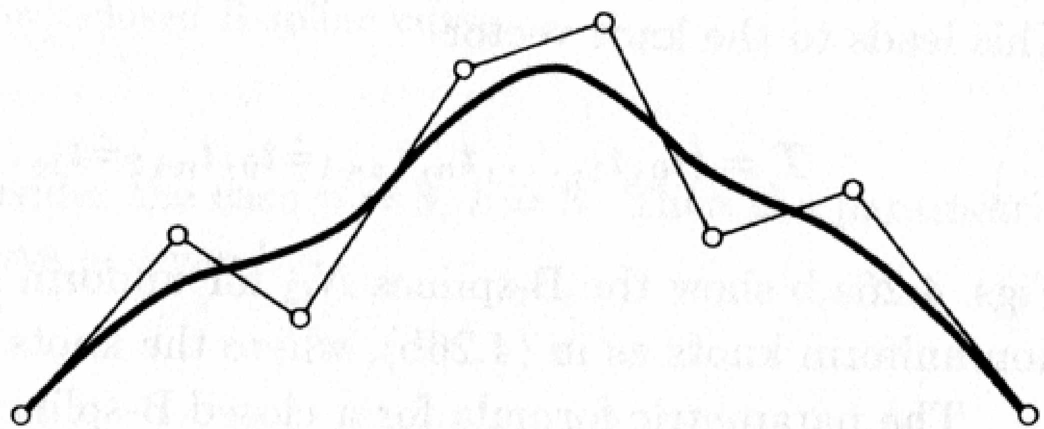


Fig. 4.25b. B-spline curve with $k = 4$, $n = 7$.



k is our d - top curve has order 3, bottom order 4

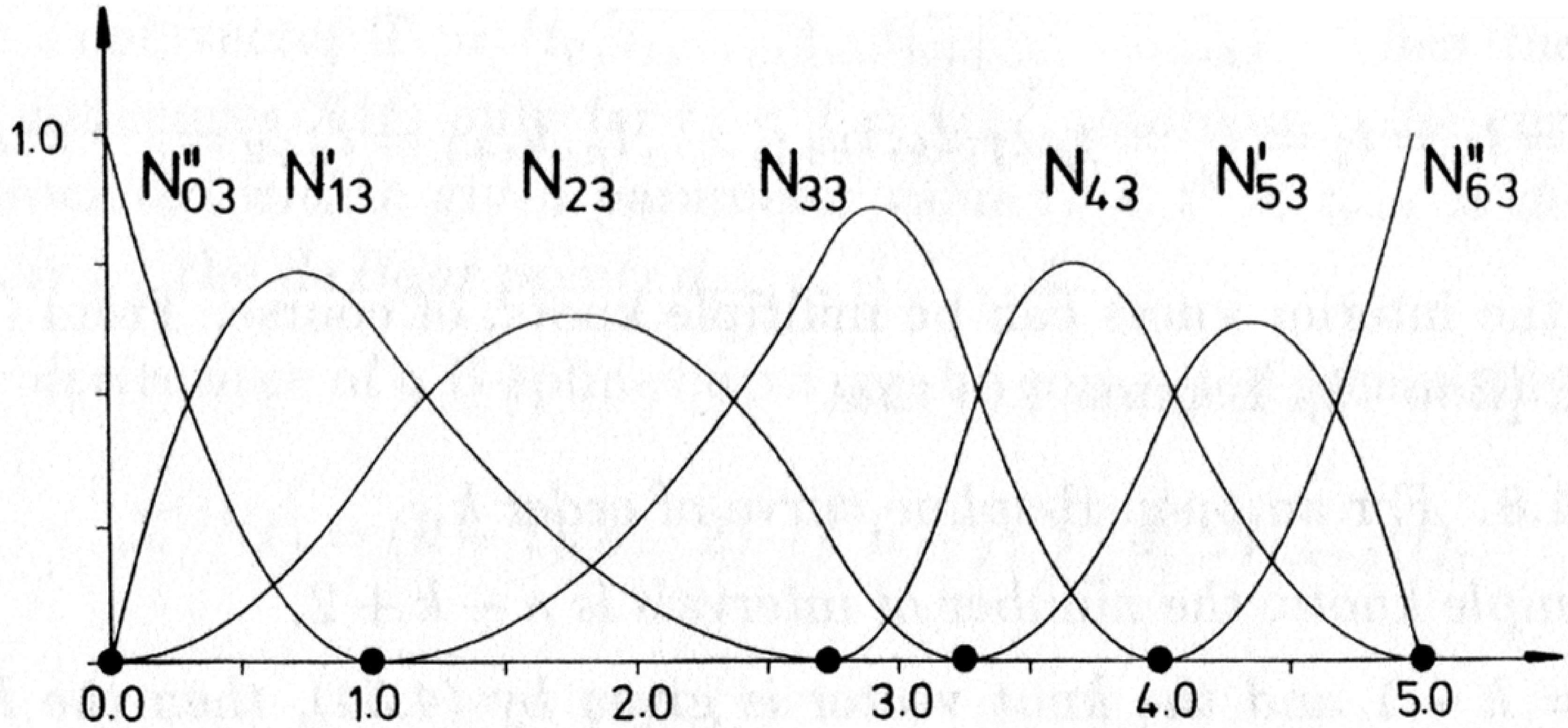


Fig. 4.24b. B-splines for an open B-spline curve with non-uniform knot vector.

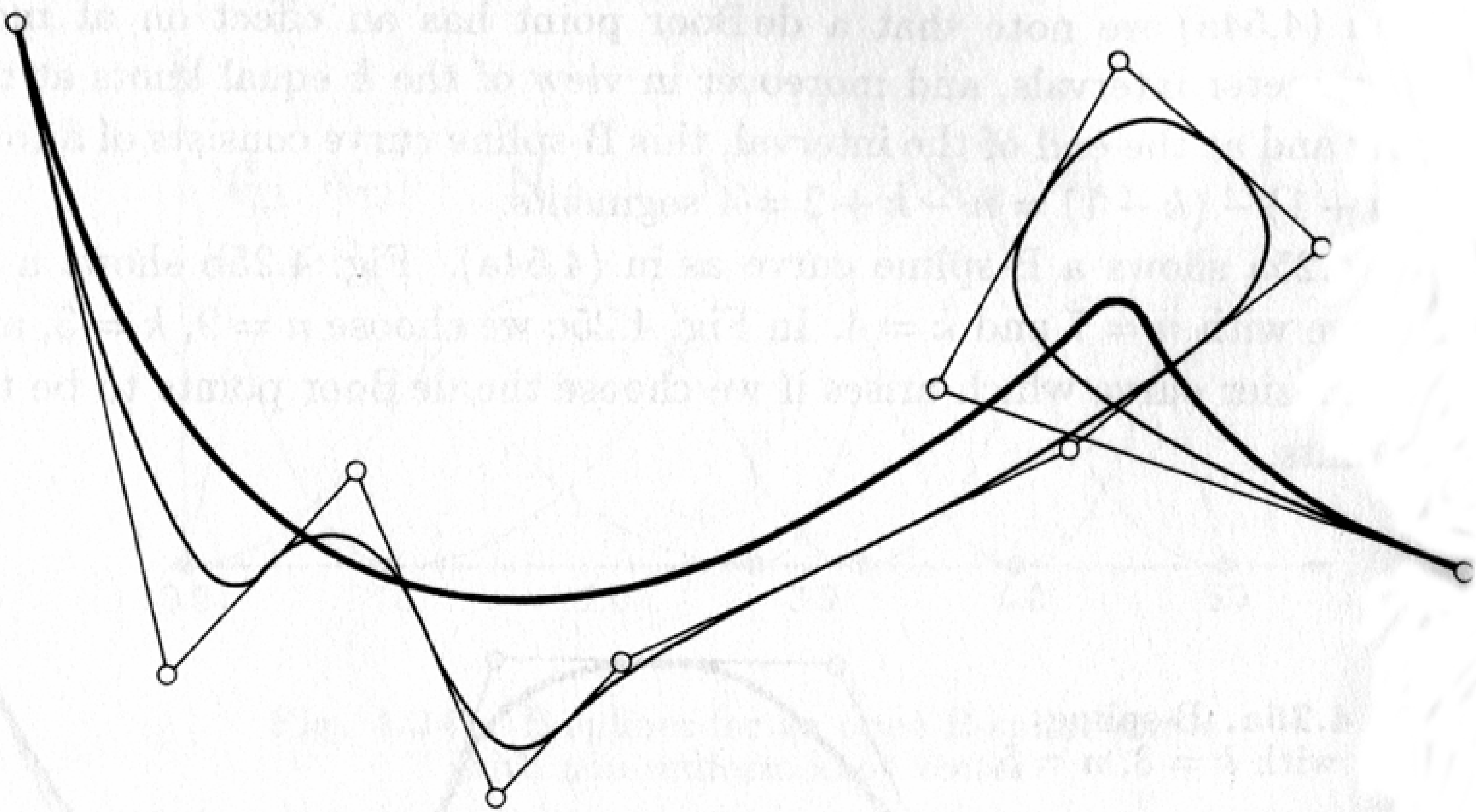


Fig. 4.25c. B-spline curve with $k = 3, n = 9$ and the Bézier curve of degree 9 with the same control polygon.

Bézier curve is the heavy curve

B-Spline properties

- For a B-spline curve of order d
  - if m knots coincide, the curve is C^{d-m-1} at the corresponding point
  - if d-1 points of the control polygon are collinear, then the curve is tangent to the polygon
  - if d points of the control polygon are collinear, then the curve and the polygon have a common segment
  - if d-1 points coincide, then the curve interpolates the common point and the two adjacent sides of the polygon are tangent to the curve
  - each segment of the curve lies in the convex hull of the associated d points

Rational Curves

$$x(t) = \frac{X(t)}{W(t)} \qquad y(t) = \frac{Y(t)}{W(t)} \qquad z(t) = \frac{Z(t)}{W(t)}$$

- X(t), etc are usual cubics
- Can represent even more curves (e.g. conics)
- Divsion by W(t) can be deferred (part of perspective transformation).

NURBS

- When the X(t) etc are B-Splines, AND, the knots are non-uniform, we get the comon choice of general, powerful, modeling approach
- NURBS= Non-Uniform, Rational, B-Splines

Surfaces

- Modelling surfaces is similar to modeling curves, just more complicated
- We have already seen special cases (sweeps, surfaces of revolution, etc)

Parametric Bicubic Surfaces (§9.3)

- Want an expression Q(s,t) analogous to Q(t) for curves.
- Consider fixed t:
  - Q(s) = GMS (our curve representation)
  - Now suppose that G varies with T
  - $Q(s,t) = [G_1(t) \;\; G_2(t) \;\; G_3(t) \;\; G_4(t)] \cdot M \cdot S$

$$Q_x(s,t) = [G_1^{(x)}(t) \quad G_2^{(x)}(t) \quad G_3^{(x)}(t) \quad G_3^{(x)}(t)] \cdot M \cdot S$$

$$G_i^{(x)}(t) = [\mathbf{g}_{i1} \quad \mathbf{g}_{i2} \quad \mathbf{g}_{i3} \quad \mathbf{g}_{i4}] \cdot M \cdot T$$

$$G_i^{(x)}(t) = G_i^{(x)}(t)^T = T^T \cdot M^T \cdot [\mathbf{g}_{i1} \quad \mathbf{g}_{i2} \quad \mathbf{g}_{i3} \quad \mathbf{g}_{i4}]^T$$

$$Q_x(s,t) = T^T \cdot M^T \cdot G_x \cdot M \cdot S$$

$$where \qquad G_x = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}$$

Applying the formula

- Hermite (§9.3.1)

$$x(s,t) = T^T \cdot M_H^T \cdot G_{H_x} \cdot M_H \cdot S$$

- Bezier (§9.3.2)

$$x(s,t) = T^T \cdot M_B^T \cdot G_{B_x} \cdot M_B \cdot S$$

- Uniform B-Spline (§9.3.3)

Rendering

- Calculating Normals (See §9.3.4)
- Still (may) need to break the surface into a grid (discussed briefly in §9.3.5).