Parametric curve normals

- Normal is cross product of tangent in t direction and s direction.

$$\left[\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right] \times \left[\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial t}\right]$$

- Cylinder: normal is cross-product of curve tangent and direction vector
- Surface of revolution: take curve normal and spin round axis

Blended curves

- Assume degree 3
- Includes Hermite, Bézier and others

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = CT = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Blended curves

- Assume degree 3
- Includes Hermite, Bézier, and others

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = [G_1 \quad G_2 \quad G_3 \quad G_4] \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Hermite

- Geometry matrix
  - First two columns are endpoints
  - Next two columns are derivatives at those points

$$M_H = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Where does this come from?

Hermite

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = [G_1 \quad G_2 \quad G_3 \quad G_4] M_H \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$x(t) = [G_{1x} \quad G_{2x} \quad G_{3x} \quad G_{4x}] M_H \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Details
Optional

Hermite

$$x(0) = [G_{1x} \quad G_{2x} \quad G_{3x} \quad G_{4x}] M_H \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x'(0) = [G_{1x} \quad G_{2x} \quad G_{3x} \quad G_{4x}] M_H \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x(1) = [G_{1x} \quad G_{2x} \quad G_{3x} \quad G_{4x}] M_H \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x'(1) = [G_{1x} \quad G_{2x} \quad G_{3x} \quad G_{4x}] M_H \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Details
Optional

Hermite

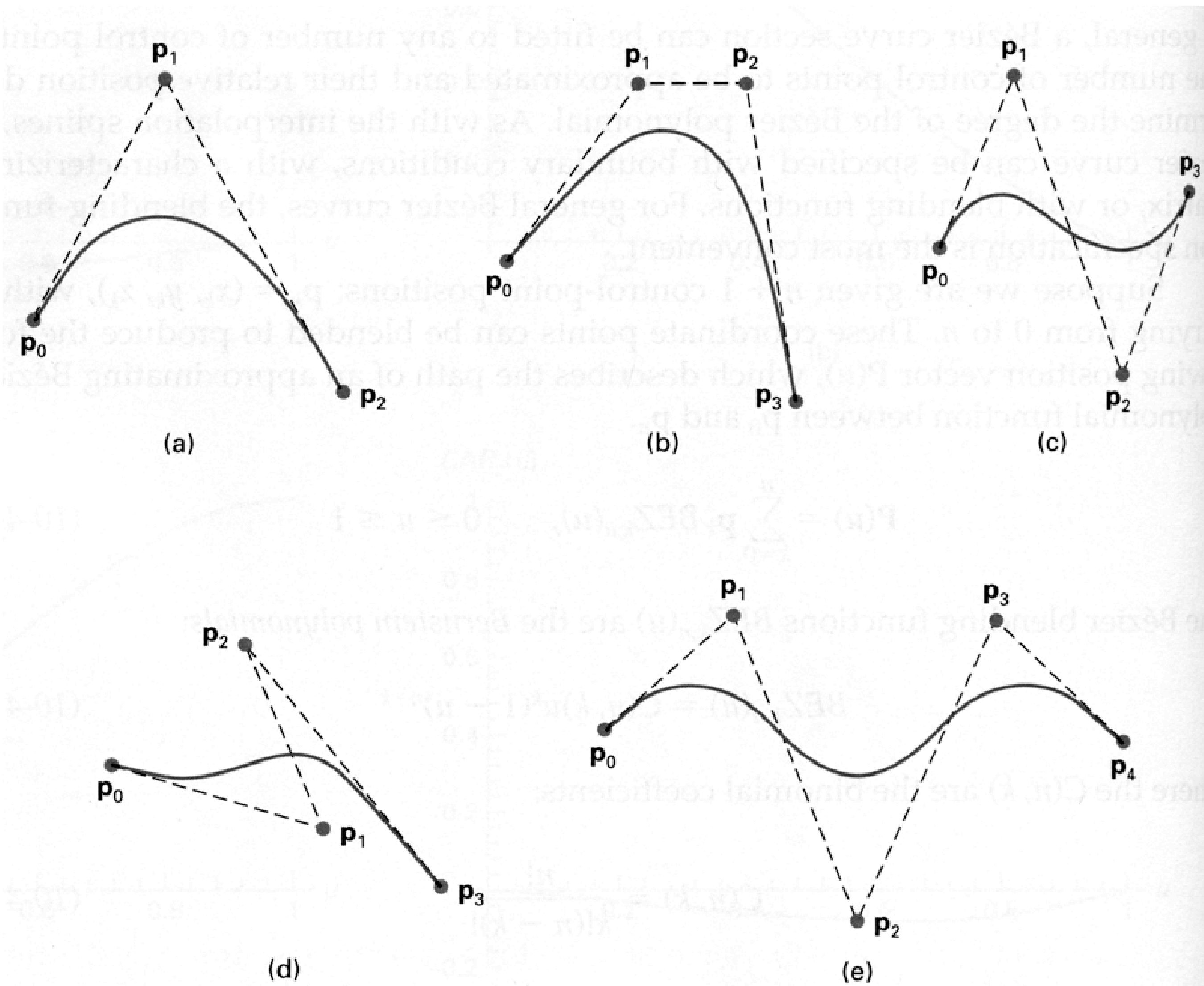
$$[P_{1x} \quad P_{4x} \quad R_{1x} \quad R_{4x}] = G_{H_x} = G_{H_x} M_H \begin{vmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$M_H = \begin{vmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Bézier

- Curve goes through two control points
- Curve is adjusted by moving two (cubic case) other control points
- Tangent at endpoints is in direction of adjacent control point
- Curve lies in convex hull of all 4 (cubic case) control points.
  - First two columns are endpoints
  - Next two columns are derivatives at those points

Example Bézier Curves



Bézier

- Geometry matrix
  - First two columns are endpoints
  - Next two are like derivatives from the Hermite case, but are now defined by

$$R_1 = 3(P_2 - P_1)$$
$$R_2 = 3(P_4 - P_3)$$

  - Note that this gives our condition on endpoint tangents
  - Factor of 3 gives good "balance" in control point effect, and is needed to be consistent with other derivations (e.g., Bernstein polynomials, subdivision, etc).

$$R_1 = 3(P_2 - P_1)$$

$$R_2 = 3(P_4 - P_3)$$

Means that

$$\begin{bmatrix} P_1 & P_4 & R_1 & R_2 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

M_{HB}

Recall Hermite

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} P_1 & P_4 & R_1 & R_2 \end{bmatrix} M_H \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

From previous slide

$$\begin{bmatrix} P_1 & P_4 & R_1 & R_2 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} M_{HB}$$

So, for Bézier

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} M_{HB} M_H \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Want M_B in $Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} M_B \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$

$$M_B = M_{HB} M_H$$

$$= \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bézier in standard form (summary)

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} M_B \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

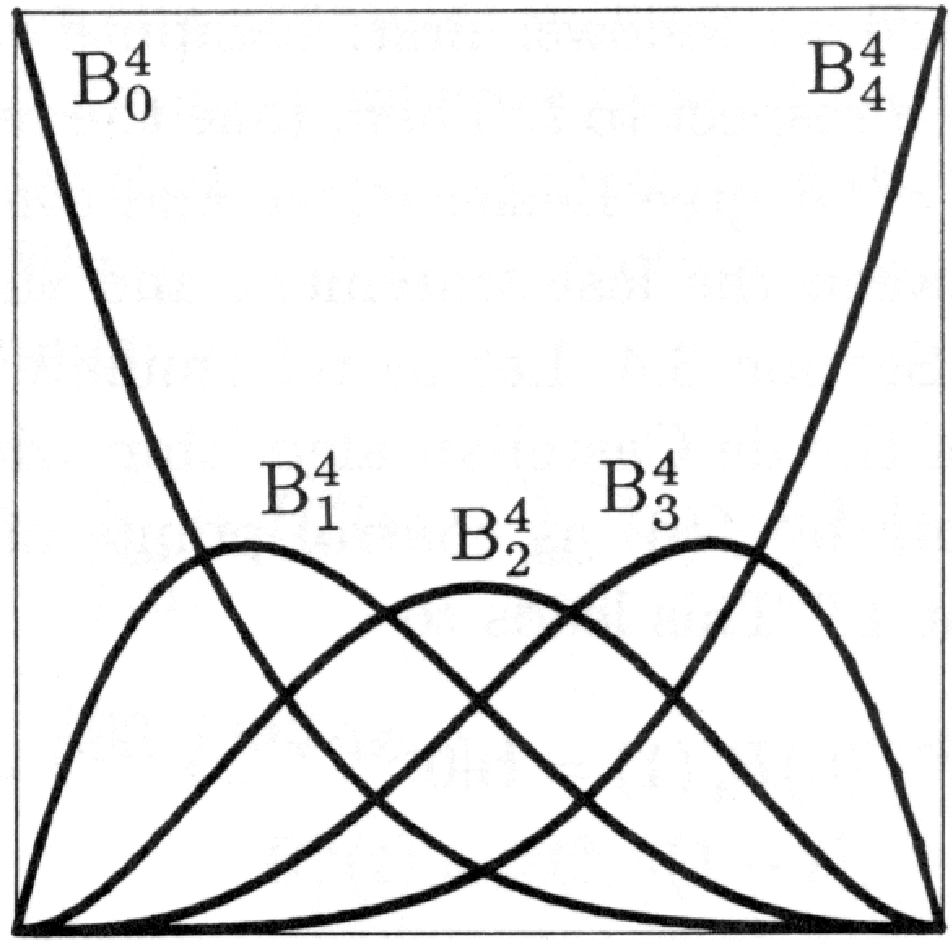
$$M_B = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bézier curves - II

- Blending functions are the Bernstein polynomials

$$c(t) = \sum_{i=0}^{n} p_i B_i^n(t)$$

$$B_i^n(t) = \binom{n}{i} t^i (1 - t)^{n-i}$$

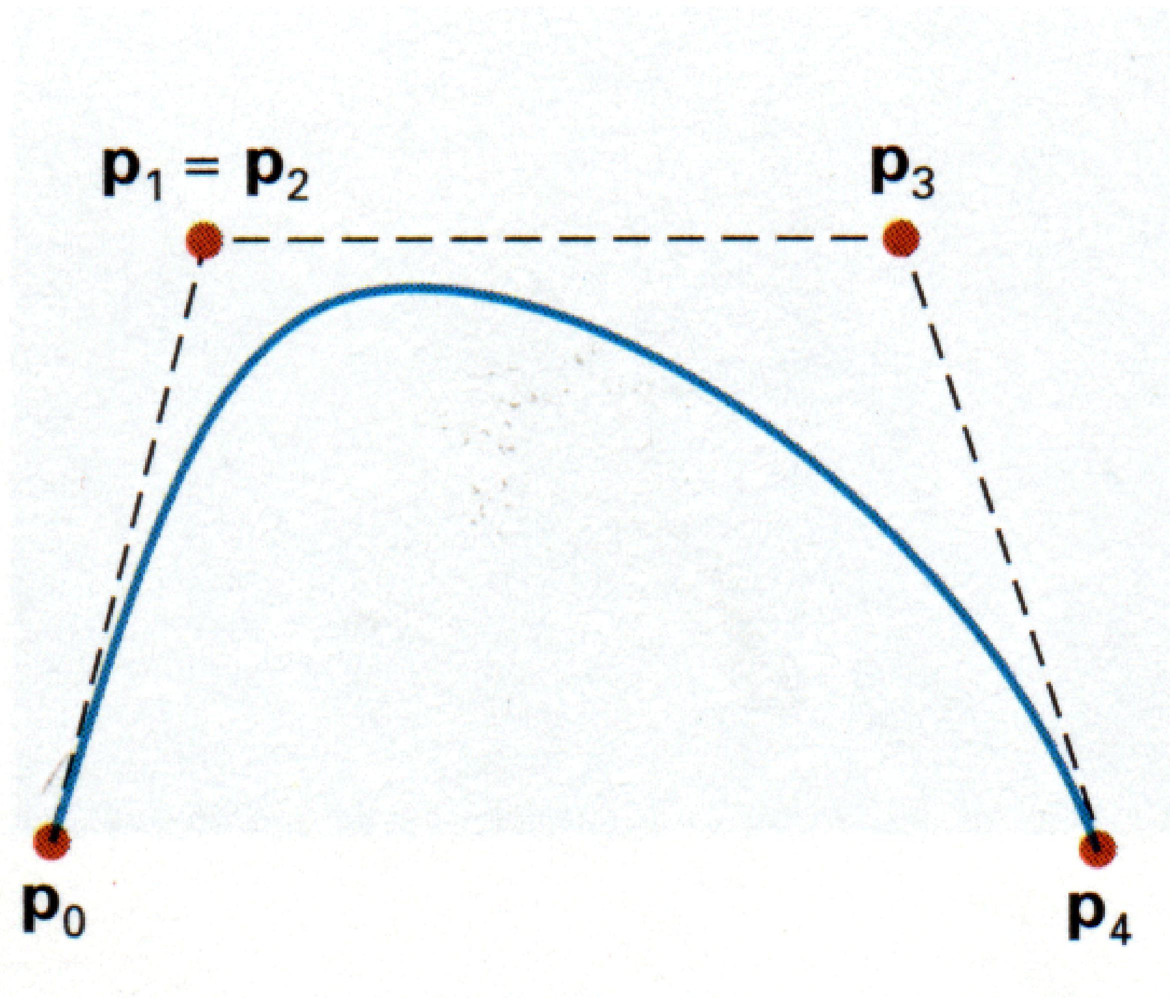


Bézier curves - III

- Bernstein polynomials have several important properties
  - they sum to 1, hence curve lies within convex hull of control points
  - curve interpolates its endpoints
  - curve's tangent at start lies along the vector from p0 to p1
  - tangent at end lies along vector from pn-1 to pn

Bézier curve tricks - I

- "Pull" a curve toward a control point by doubling the control point



Bézier curve tricks-II

- Close the curve by making last point and first point coincident
  - curve has continuous tangent if first segment and last segment are collinear