Interpolating Splines

- Key idea:
  - high degree interpolates are badly behaved->
  - construct curves out of low degree segments

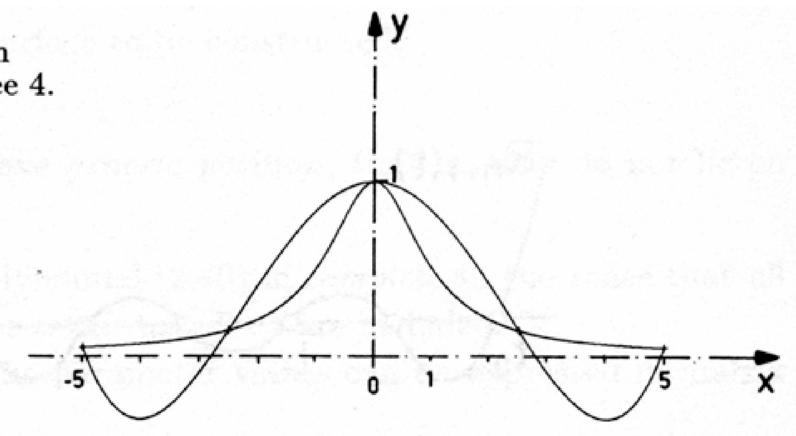


Fig 2.16a. Interpolation by a polynomial of degree 4.

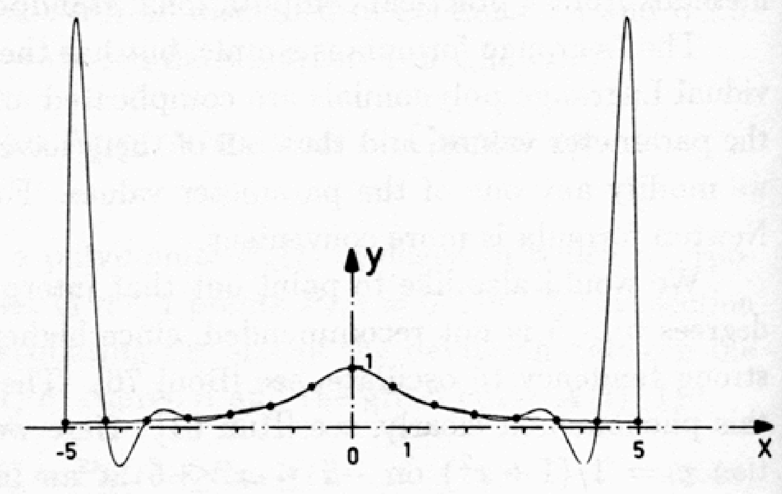


Fig 2.16c. Interpolation by a polynomial of degree 14.

Interpolating Splines - II

- n+1 points;
- write derivatives X’
- X_i is spline for interval between P_i and P_{i+1}

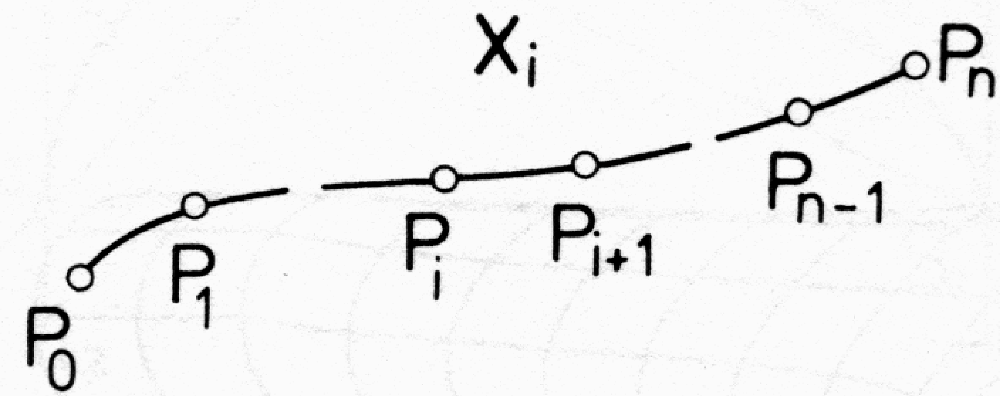


Fig. 3.11. The spline segment $\boldsymbol{X}_i$.

Interpolating Splines - II

- Bolt together a series of Hermite curves with derivatives matching at joints (Knots).
- But where are the derivative values to come from?
  - Measurements
  - Combination of points (see cardinal splines--next topic)
  - Continuity considerations
  - Conventions for enpoints

- Cardinal splines

Equation Optional

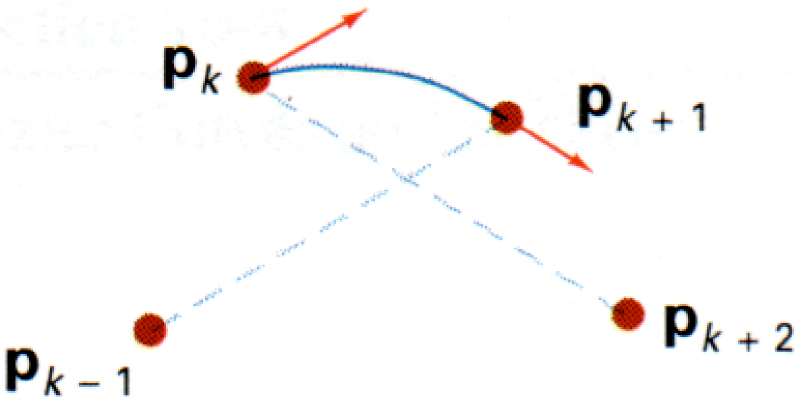
$$P'_k = \left(\frac{1}{2}\right)(1 - t)\left(P_{k+1} - P_{k-1}\right)$$

  - t is “tension”
  - still need to specify endpoint tangents
    - or use difference between first two, last two points

(Don’t confuse t with parameter!)

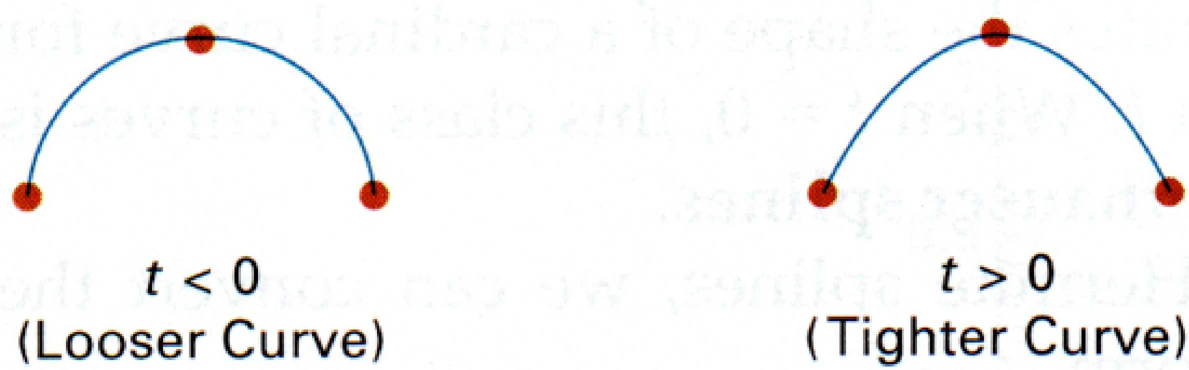
Details Optional

Tension



- larger values of tension give tighter curves (limit (as t-->1) is linear interpolate).

(Don't confuse t with parameter!)



Interpolating Splines

- Intervals:

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b.$$

$$\Delta t_i := t_{i+1} - t_i.$$

  - t values often called "knots"

- Spline form:

$$\boldsymbol{X}_i(t) := \boldsymbol{A}_i(t - t_i)^3 + \boldsymbol{B}_i(t - t_i)^2 + \boldsymbol{C}_i(t - t_i) + \boldsymbol{D}_i,$$
$$t \in [t_i, t_{i+1}], \qquad i = 0(1)N\text{–}1,$$

Continuity

- Require at endpoints:
  - endpoints equal
  - 1'st derivatives equal
  - 2'nd derivatives equal
- Now we get extra information from continuity (instead of tension equation, tangent measurements, etc)

$$\boldsymbol{X}_i(t_i) = \boldsymbol{X}_{i-1}(t_i) \quad \text{or} \quad \boldsymbol{X}_i(t_{i+1}) = \boldsymbol{X}_{i+1}(t_{i+1}),$$
$$\boldsymbol{X}'_i(t_i) = \boldsymbol{X}'_{i-1}(t_i) \quad \text{or} \quad \boldsymbol{X}'_i(t_{i+1}) = \boldsymbol{X}'_{i+1}(t_{i+1}),$$
$$\boldsymbol{X}''_i(t_i) = \boldsymbol{X}''_{i-1}(t_i) \quad \text{or} \quad \boldsymbol{X}''_i(t_{i+1}) = \boldsymbol{X}''_{i+1}(t_{i+1}).$$

Details
Optional

- From endpoint and 1'st derivative:

$$X_i(t_i) = P_i = D_i, \qquad X_i(t_{i+1}) = P_{i+1} = A_i \Delta t_i^3 + B_i \Delta t_i^2 + C_i \Delta t_i + D_i,$$
$$X_i'(t_i) = P_i' = C_i, \qquad X_i'(t_{i+1}) = P_{i+1}' = 3A_i \Delta t_i^2 + 2B_i \Delta t_i + C_i,$$

- So that

$$A_i = \frac{1}{(\Delta t_i)^3}[2(P_i - P_{i+1}) + \Delta t_i(P_i' + P_{i+1}')],$$
$$B_i = \frac{1}{(\Delta t_i)^2}[3(P_{i+1} - P_i) - \Delta t_i(2P_i' + P_{i+1}')].$$

- Yielding:

$$X_i(t) =$$
$$P_i\left(2\frac{(t-t_i)^3}{(\Delta t_i)^3} - 3\frac{(t-t_i)^2}{(\Delta t_i)^2} + 1\right) + P_{i+1}\left(-2\frac{(t-t_i)^3}{(\Delta t_i)^3} + 3\frac{(t-t_i)^2}{(\Delta t_i)^2}\right)$$
$$+ P_i'\left(\frac{(t-t_i)^3}{(\Delta t_i)^2} - 2\frac{(t-t_i)^2}{\Delta t_i} + (t-t_i)\right) + P_{i+1}'\left(\frac{(t-t_i)^3}{(\Delta t_i)^2} - \frac{(t-t_i)^2}{\Delta t_i}\right)$$

Details
Optional

- Second Derivative:

$$X_i''(t) = 6P_i\left(\frac{2(t-t_i)}{(\Delta t_i)^3} - \frac{1}{(\Delta t_i)^2}\right) + 6P_{i+1}\left(-2\frac{(t-t_i)}{(\Delta t_i)^3} + \frac{1}{(\Delta t_i)^2}\right)$$
$$+ 2P_i'\left(3\frac{(t-t_i)}{(\Delta t_i)^2} - \frac{2}{\Delta t_i}\right) + 2P_{i+1}'\left(\frac{3(t-t_i)}{(\Delta t_i)^2} - \frac{1}{\Delta t_i}\right).$$

- Want:

$$X_{i-1}''(t_i) = X_i''(t_i)$$

- Yielding:

$$\Delta t_i P_{i-1}' + 2(\Delta t_{i-1} + \Delta t_i)P_i' + \Delta t_{i-1} P_{i+1}'$$
$$= 3\frac{\Delta t_{i-1}}{\Delta t_i}(P_{i+1} - P_i) + 3\frac{\Delta t_i}{\Delta t_{i-1}}(P_i - P_{i-1}).$$

Missing equations

- Recurrence relations represent d(n-1) equations in d(n+1) unknowns (d is dimension)
- We need to supply the derivative at the start and at the finish (or two equivalent constraints)
- Options:
  - second derivatives vanish at each end (natural spline)
  - give slopes at the boundary
    - vector from first to second, second last to last
    - parabola through first three, last three points
  - third derivative is the same at first, last knot

B-splines - I

- Now consider stitching together curves which do not necessarily pass through the control points.
- Local control
- Blending functions are non-zero over limited range--thus they are like “switches”
- In the simplest case of uniformly spaced control points, the blending functions will be shifted versions of the same function.

B-splines - II

- Curve (general case):

$$X(t) = \sum_{k=0}^{n} P_k B_{k,d}(t)$$

- The “order” d is:

$$2 \le d \le n+1$$

- Usual case: n is 4, d is 3.

B-Spline Blending Functions

- Knots
  - parameter values where curve segments meet, as in Hermite example

$$\left(t_0, t_1, \ldots, t_{n+d}\right)$$

where $t_0 \le t_1 \le \ldots \le t_{n+d}$

- Blending functions

$$B_{k,1}(t) = \begin{cases} 1 & t_k \le t < t_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k}\right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}}\right) B_{k+1,d-1}(t)$$

- Division by 0 gives 0

B-Spline Blending Functions

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k}\right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}}\right) B_{k+1,d-1}(t) \qquad B_{k,1}(t) = \begin{cases} 1 & t_k \le t < t_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

So, assuming uniformly spaced knots,

$$B_{k,2}(t) = \quad \textbf{?}$$

B-Spline Blending Functions

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k}\right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}}\right) B_{k+1,d-1}(t)$$

$$B_{k,1}(t) = \begin{cases} 1 & t_k \le t < t_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

So, assuming uniformly spaced knots,

$$B_{k,2}(t) = \begin{cases} \left(\frac{t - t_k}{\Delta t}\right) & t_k \le t < t_{k+1} \\ \left(\frac{t_k - t}{\Delta t}\right) & t_{k+1} \le t < t_{k+2} \\ 0 & \text{otherwise} \end{cases}$$

B-Spline Blending Functions

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k}\right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}}\right) B_{k+1,d-1}(t)$$

$$B_{k,2}(t) = \begin{cases} \left(\frac{t - t_k}{\Delta t}\right) & t_k \le t < t_{k+1} \\ \left(\frac{t_k - t}{\Delta t}\right) & t_{k+1} \le t < t_{k+2} \\ 0 & \text{otherwise} \end{cases}$$

So, assuming uniformly spaced knots,

$$B_{k,3}(t) = \quad ?$$

B-Spline Blending Functions

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k}\right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}}\right) B_{k+1,d-1}(t)$$

$$B_{k,2}(t) = \begin{cases} \left(\frac{t - t_k}{\Delta t}\right) & t_k \le t < t_{k+1} \\ \left(\frac{t_k - t}{\Delta t}\right) & t_{k+1} \le t < t_{k+2} \\ 0 & \text{otherwise} \end{cases}$$

So, assuming uniformly spaced knots,

$$B_{k,3}(t) = \begin{cases} \text{quadratic function} & t_k \le t < t_{k+1} \\ \text{quadratic function} & t_{k+1} \le t < t_{k+2} \\ \text{quadratic function} & t_{k+2} \le t < t_{k+3} \\ 0 & \text{otherwise} \end{cases}$$

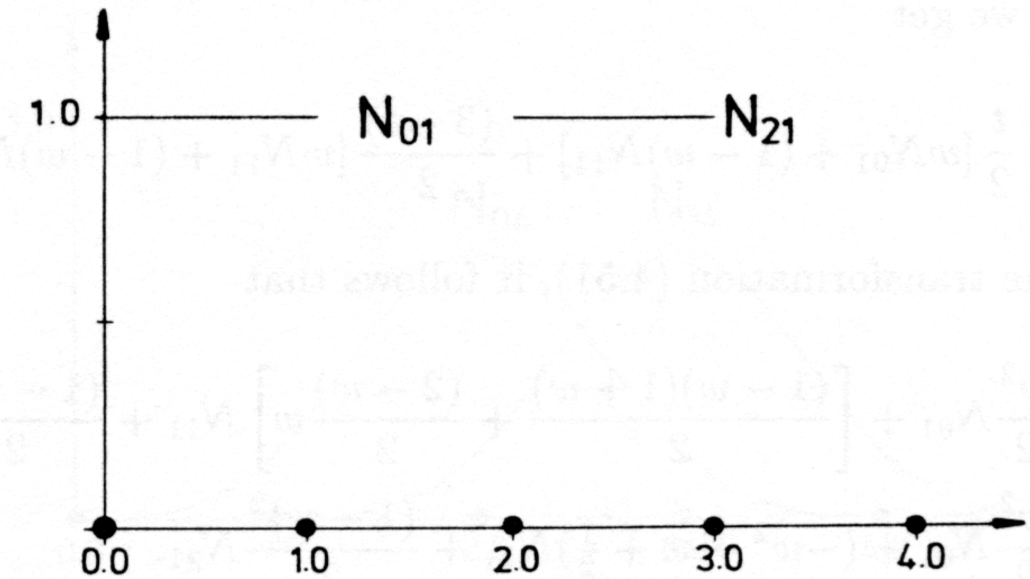


Fig. 4.22c. The B-splines N_{01}, N_{21}.

These figures show blending functions with a uniform knot vector, knots at 0, 1, 2, etc. Note that N is the same as our B

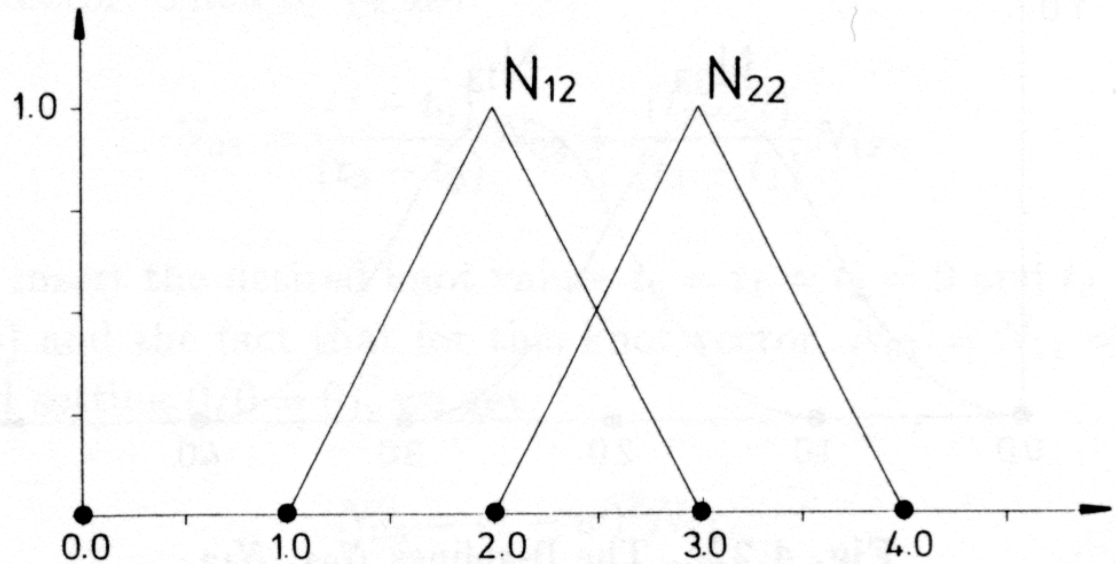


Fig. 4.22d. The B-splines N_{12}, N_{22}.

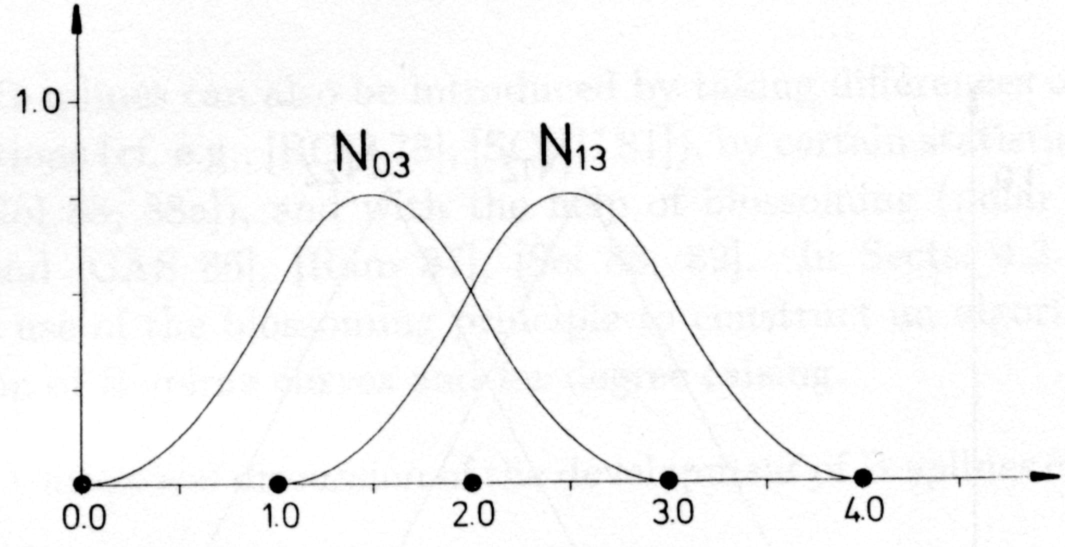
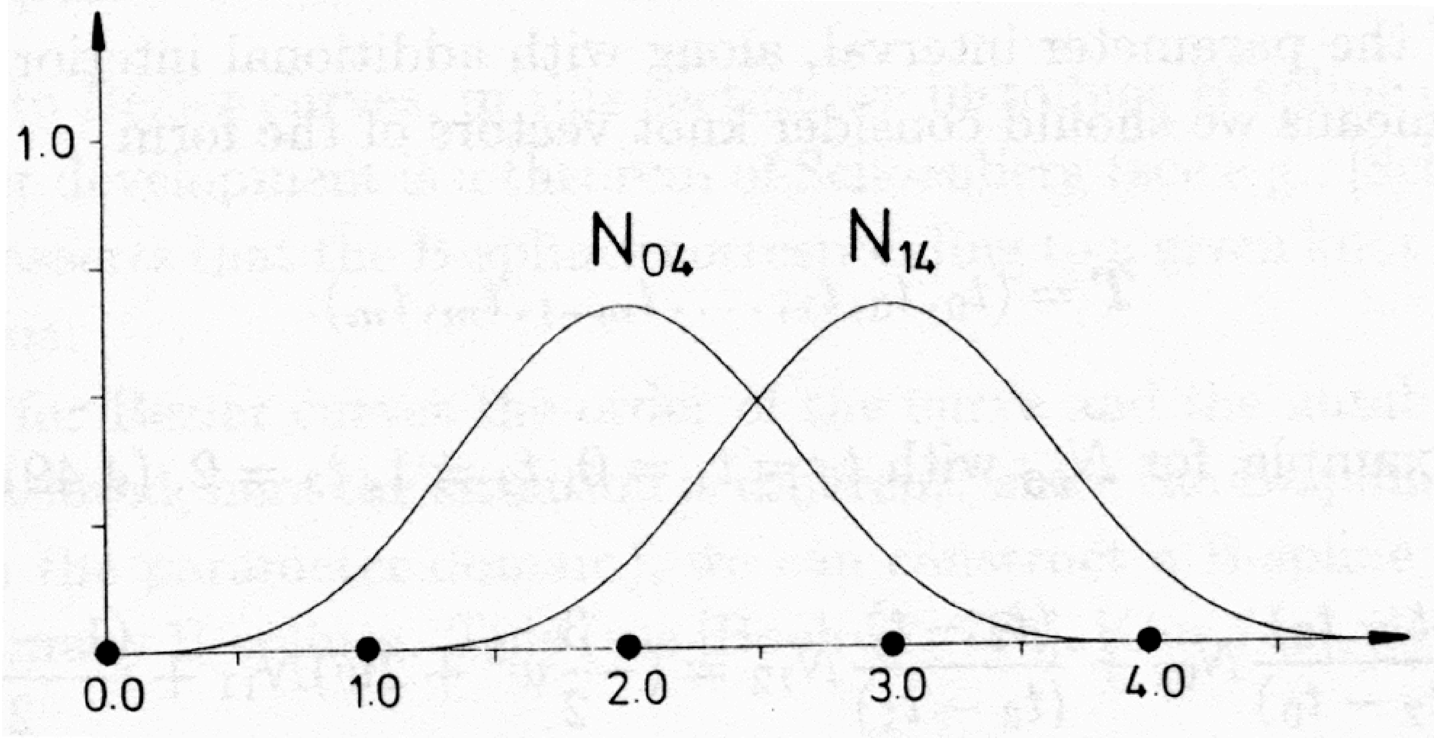


Fig. 4.22e. The B-splines N_{03}, N_{13}.



Matrix form of Uniform Cubic B-Spline Blending Functions

$$M_B = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

Closed B-Splines

- Periodically extend the control points and the knots

$$P_{n+1} = P_0$$

$$t_{n+1} = t_0$$

Fig. 4.26a.

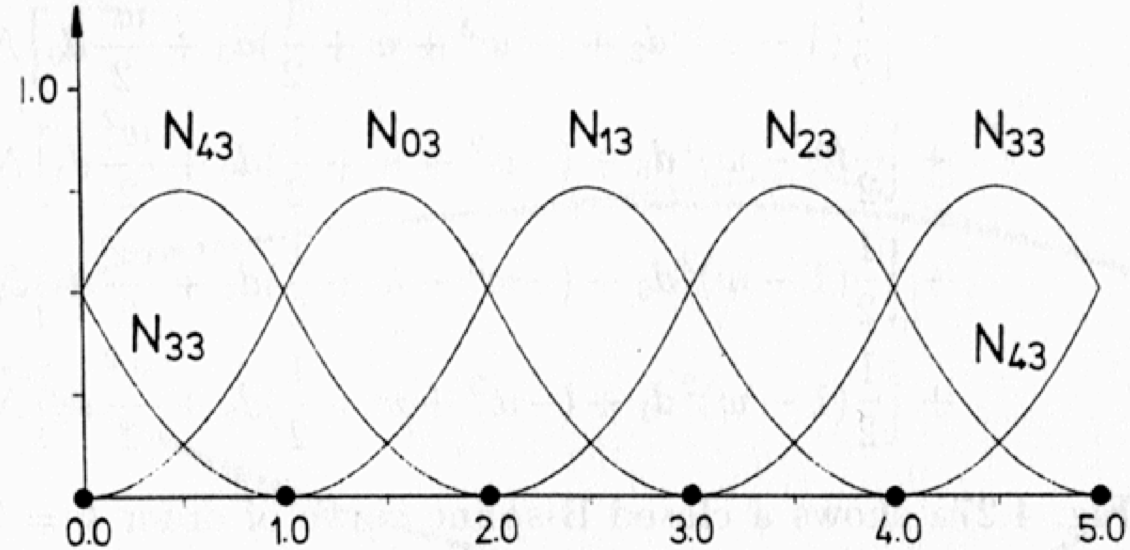


Fig. 4.26b.

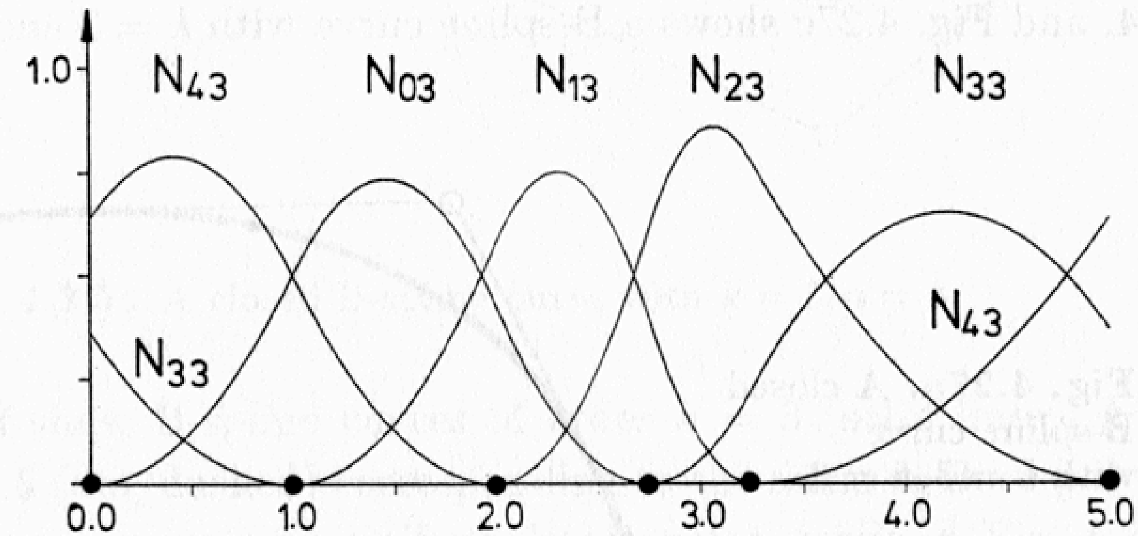


Fig. 4.26. B-splines with uniform and non-uniform knot vectors for a closed B-spline curve.

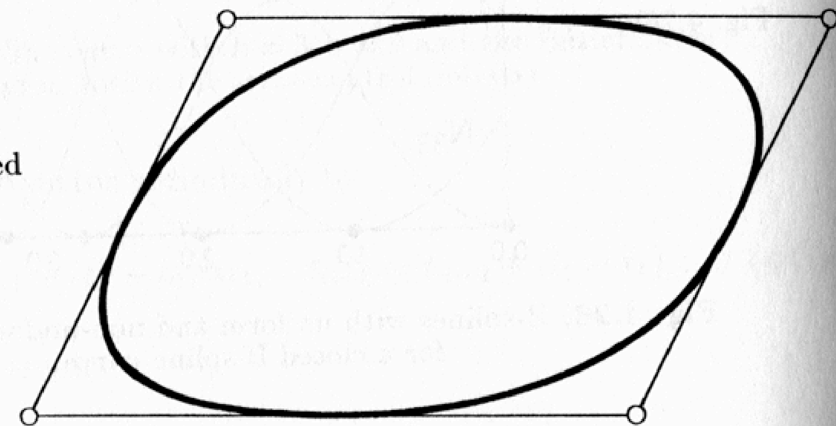


Fig. 4.27a. A closed B-spline curve with $k = 3, n = 3$.

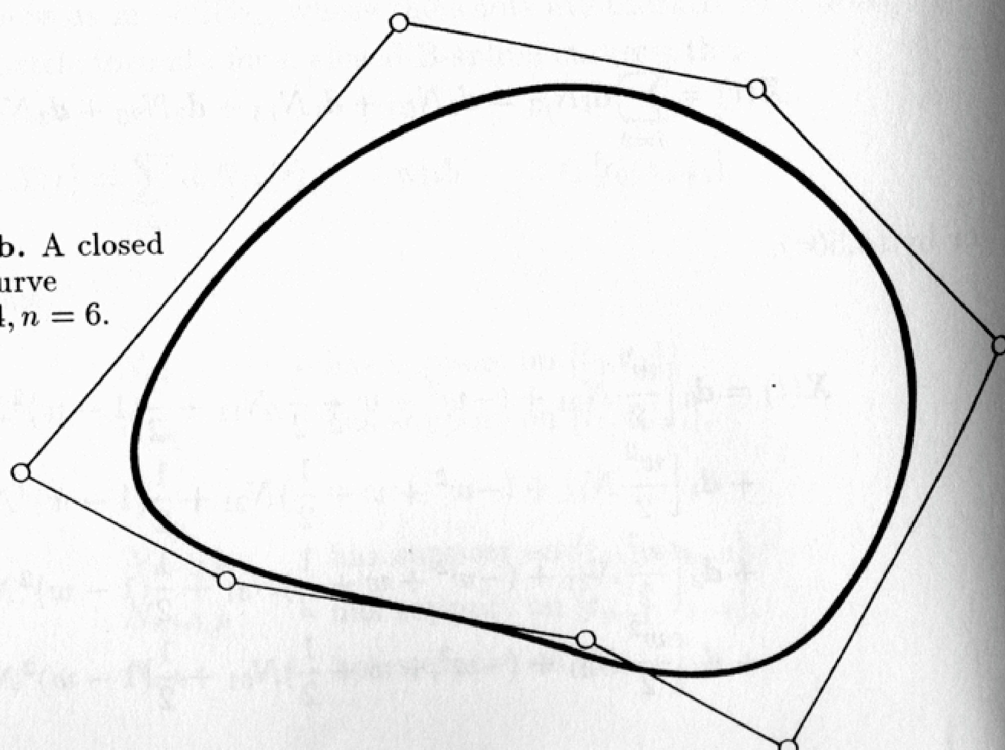


Fig. 4.27b. A closed B-spline curve with $k = 4, n = 6$.

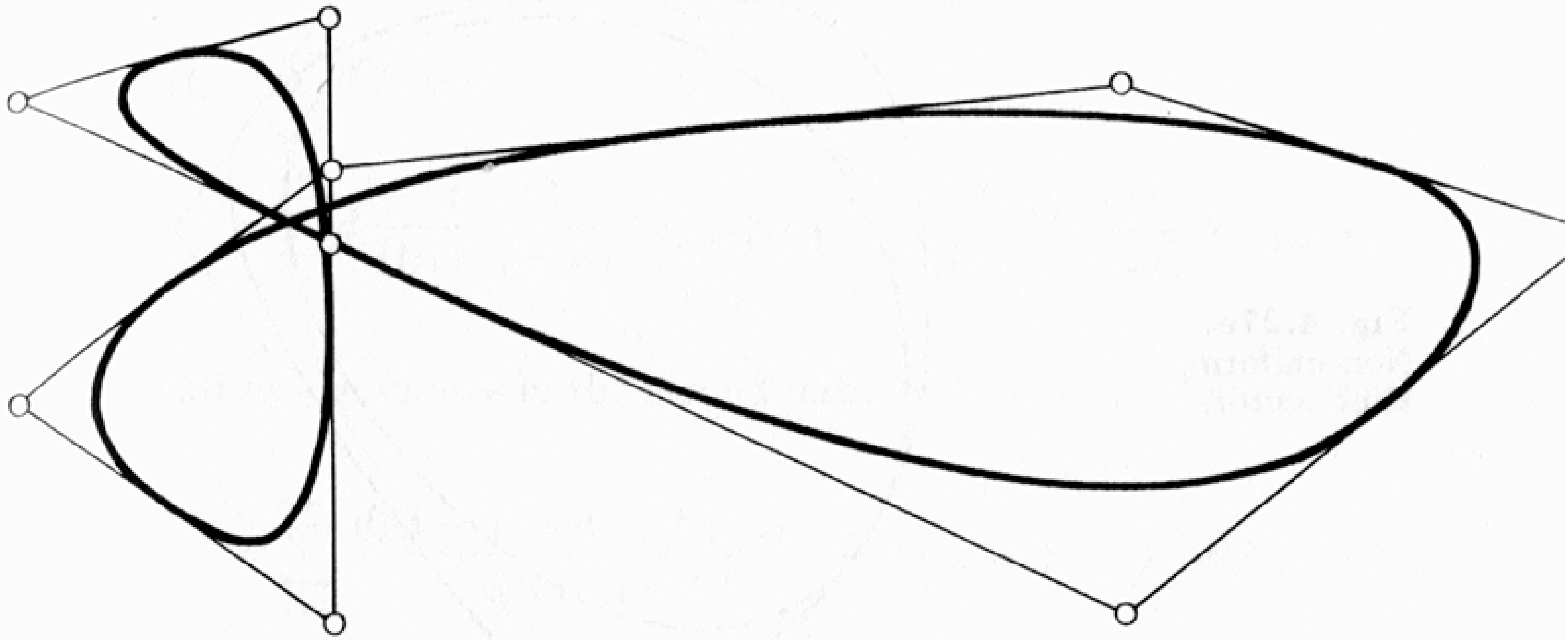
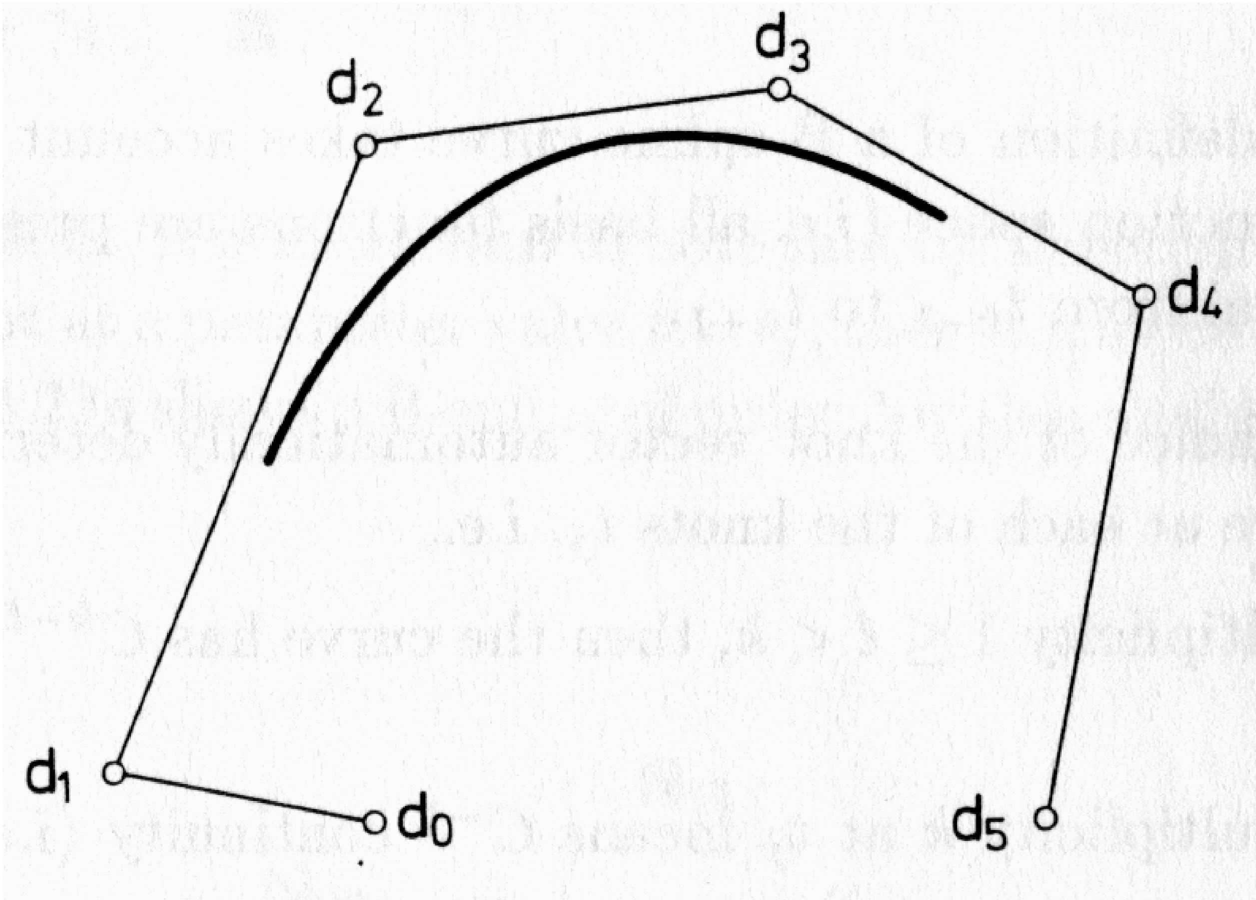


Fig. 4.27c. A closed B-spline curve with $k = 3, n = 8$.



A B-spline curve, with knots at 0,1,... and order 5

Repeated knots

- Definition works for repeated knots (if we are understanding about 0/0)
- Repeated knot reduces continuity. A B-spline blending function has continuity C^{d-2}; if the knot is repeated m times, continuity is now C^{d-m-1}
- e.g. -> quadratic B-spline (i.e. order 3) with a double knot

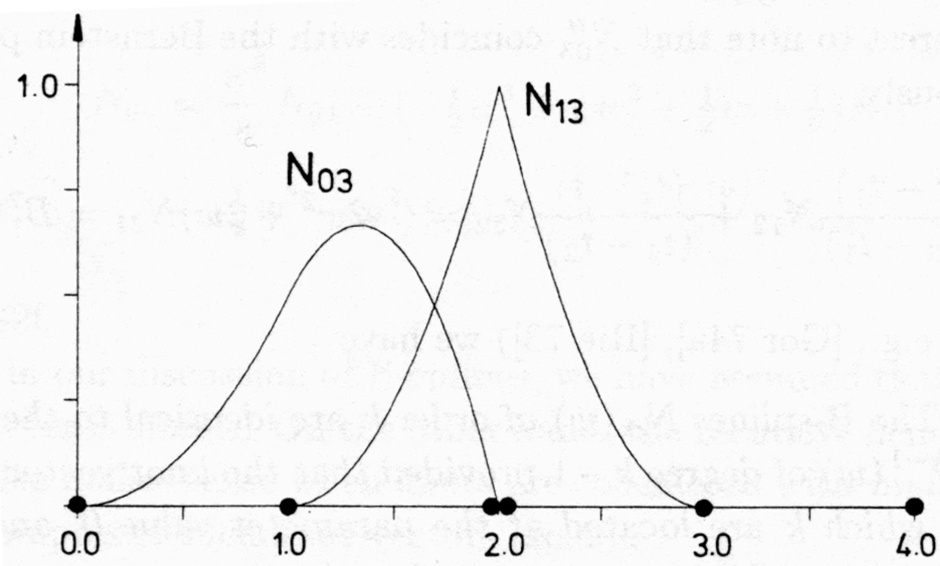


Fig. 4.22g. A quadratic B-spline with a double knot.

Most useful case

- Select the first d and the last d knots to be the same
  - we then get the first and last points lying on the curve
  - also, the curve is tangent to the first and last segment
- E.g. cubic case below
- Notice that a control point influences at most d parameter intervals - **local control**

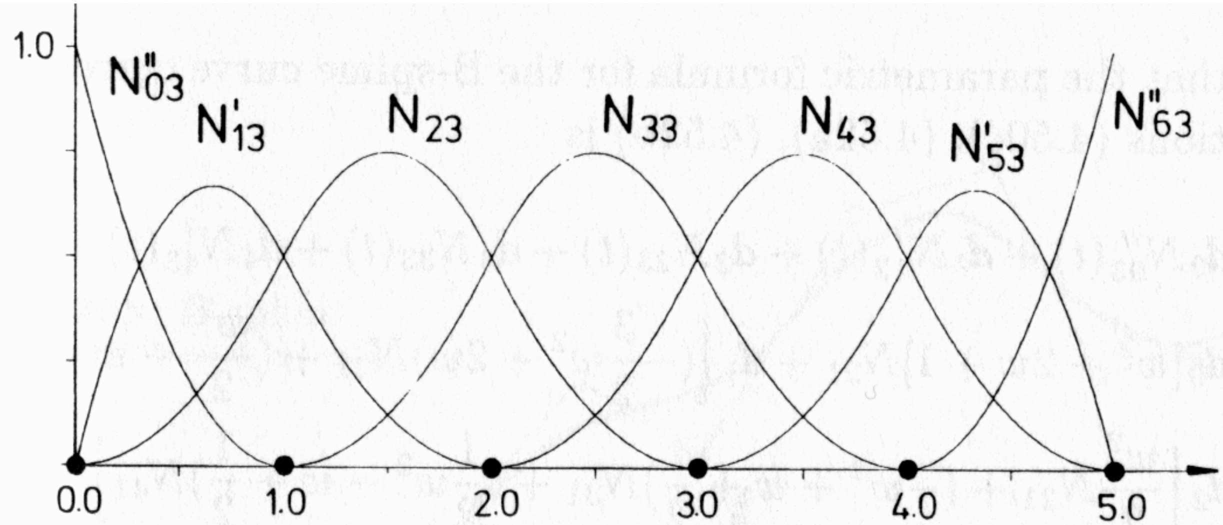


Fig. 4.24a. B-splines for an open B-spline curve with uniform knot vector.

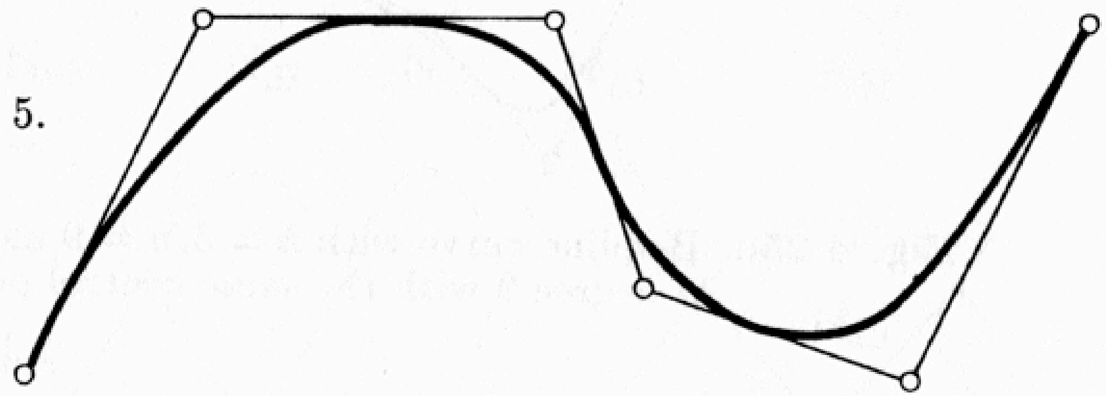


Fig. 4.25a. B-spline curve with $k = 3$, $n = 5$.

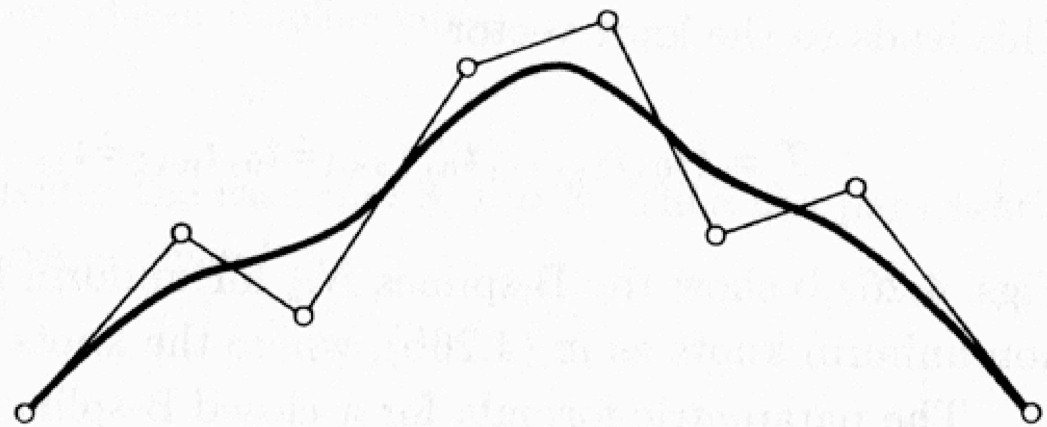


Fig. 4.25b. B-spline curve with $k = 4$, $n = 7$.

k is our d - top curve has order 3, bottom order 4

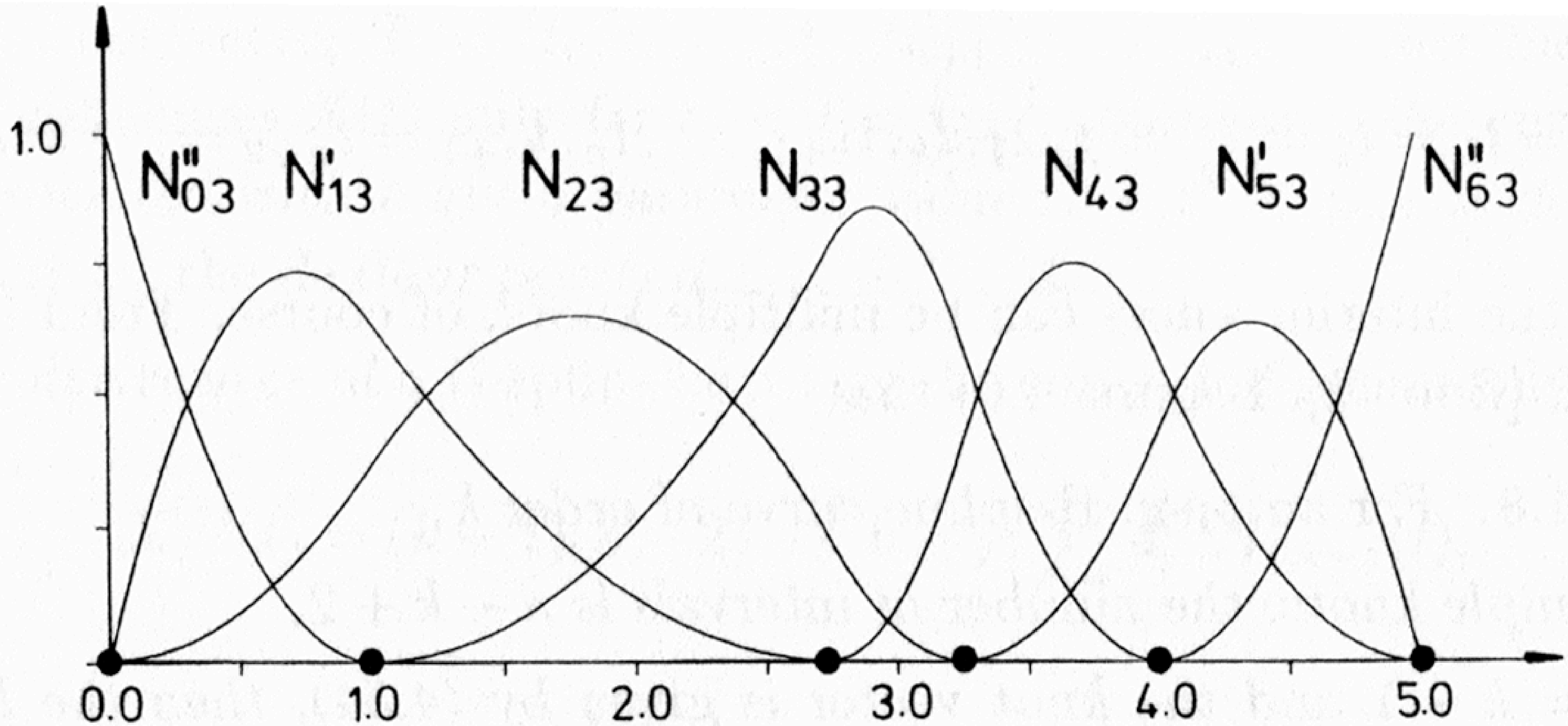


Fig. 4.24b. B-splines for an open B-spline curve with non-uniform knot vector.

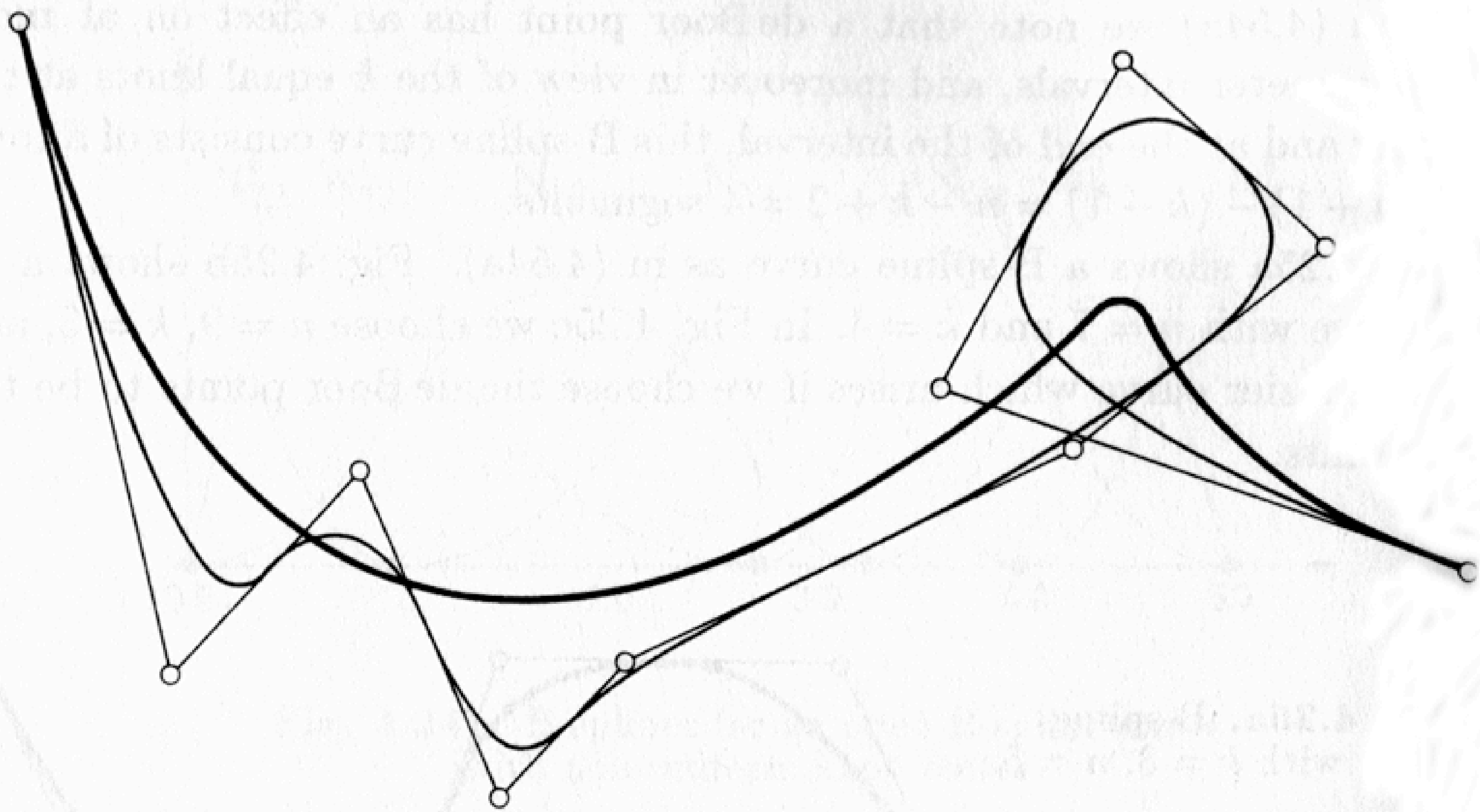


Fig. 4.25c. B-spline curve with $k = 3, n = 9$ and the Bézier curve of degree 9 with the same control polygon.

Bézier curve is the heavy curve

B-Spline properties

- For a B-spline curve of order d
  - if m knots coincide, the curve is C^{d-m-1} at the corresponding point
  - if d-1 points of the control polygon are collinear, then the curve is tangent to the polygon
  - if d points of the control polygon are collinear, then the curve and the polygon have a common segment
  - if d-1 points coincide, then the curve interpolates the common point and the two adjacent sides of the polygon are tangent to the curve
  - each segment of the curve lies in the convex hull of the associated d points