Rotations in 3D

- 3 degrees of freedom
- Det(R)=1
- Orthogonal
- Many representations are possible.
- Our representation: rotate about coordinate axes in sequence.
- Sequence of axes is arbitrary, but choice does affect the angles used (cannot use same angles with different order).
- Sign of rotation follows the Right Hand Rule--point thumb along axis in direction of increasing ordinate--then fingers curl in the direction of positive rotation).

Rotations in 3D

- About x-axis

$$M = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Rotations in 3D

- About y-axis

$$M = \begin{vmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Rotations in 3D

- About z-axis

$$M = \begin{vmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Rotations in 3D

- About X axis

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

- 90 degrees about X axis?

Rotations in 3D

- About X axis

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

- 90 degrees about X axis

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Rotations in 3D

- About Y axis

$$\begin{vmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

- 90 degrees about Yaxis?

Rotations in 3D

- About Y axis

$$\begin{vmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

- 90 degrees about Y axis

$$\begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Rotations in 3D

- 90 degrees about X then Y

$$\begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = ?$$

Y rot X rot

Rotations in 3D

- 90 degrees about X then Y

$$\begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Y rot   X rot

Rotations in 3D

- 90 degrees about X then Y

$$\begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Y rot X rot

- 90 degrees about Y then X

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = ?$$

X rot Y rot

Rotations in 3D

- 90 degrees about X then Y

$$\begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Y rot X rot

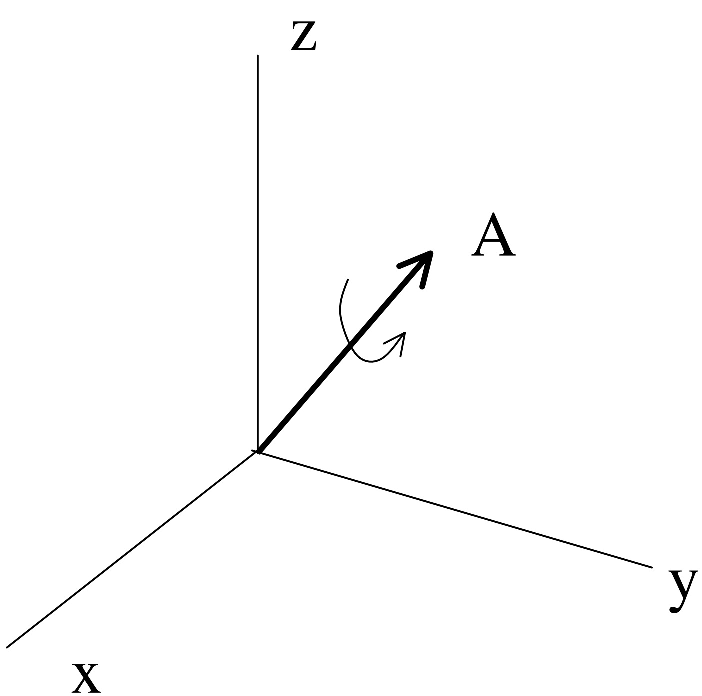
$\neq$

- 90 degrees about Y then X

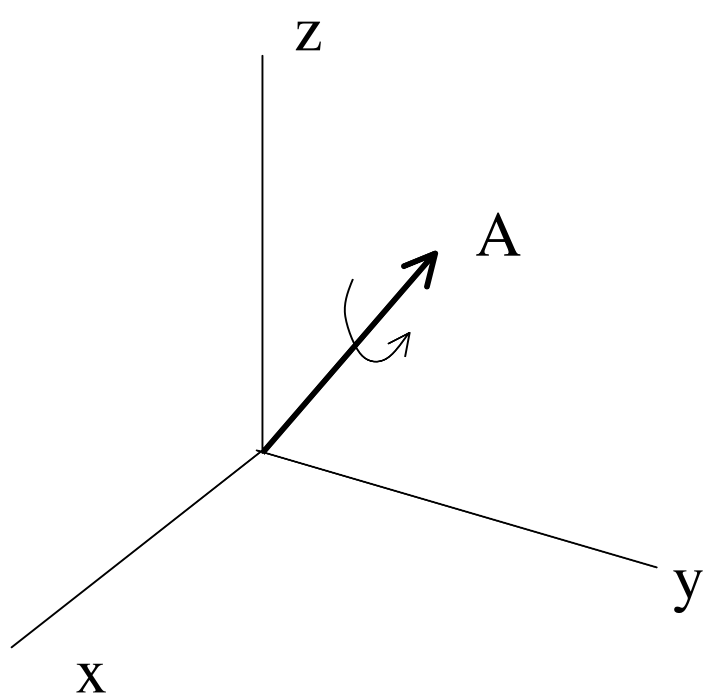
$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

X rot Y rot

Rotation about an arbitrary axis

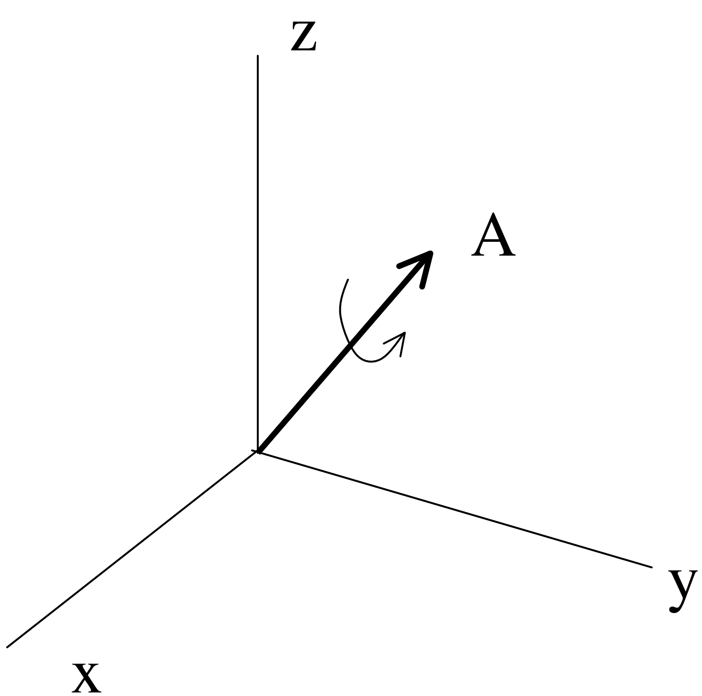


Rotation about an arbitrary axis



Strategy--rotate A to Z axis, rotate about Z axis, rotate Z back to A.

Rotation about an arbitrary axis

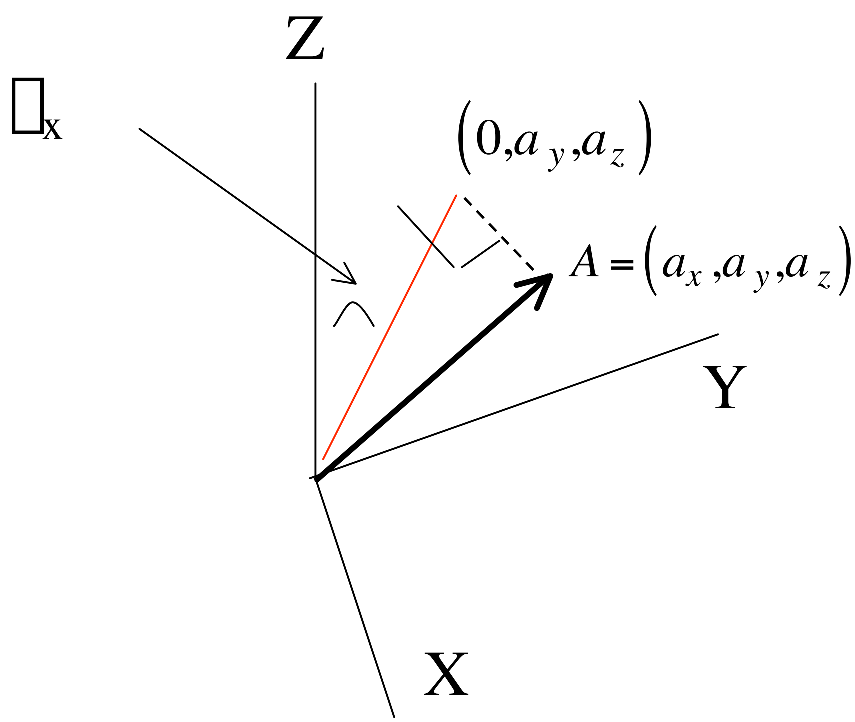


Tricky part:
rotate A to Z axis

Two steps.

1) Rotate about x to xz plane
2) Rotate about y to Z axis.

Rotation about an arbitrary axis

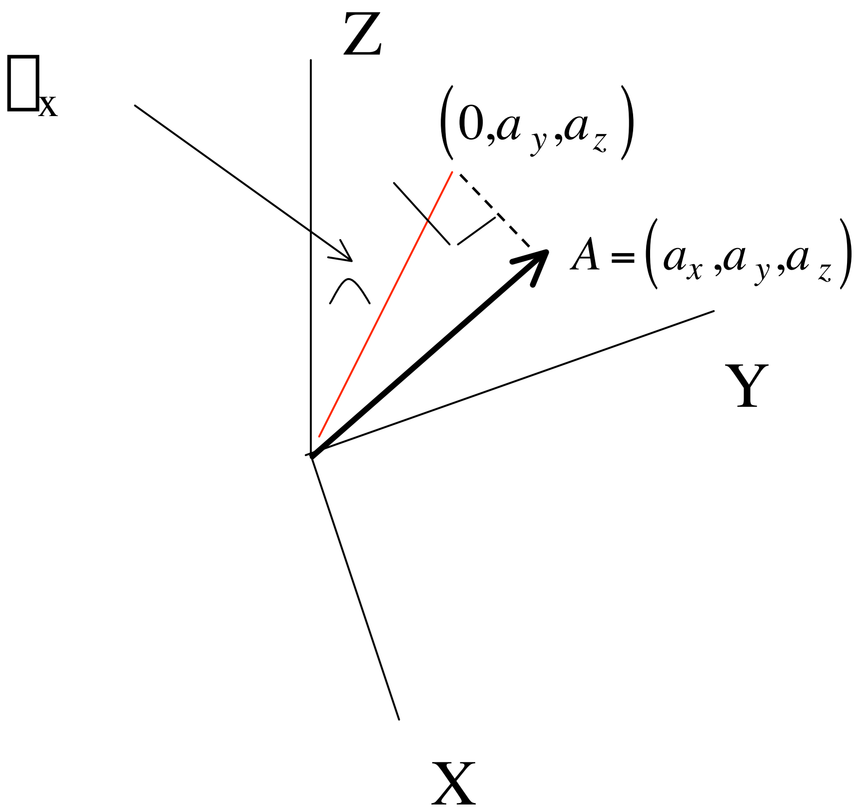


Tricky part:
rotate A to Z axis

Two steps.
1) Rotate about X to xz plane
2) Rotate about Y to Z axis.

As A rotates into the xz plane, its projection onto the YZ plane (red line) rotates through the same angle which is easily calculated.

Rotation about an arbitrary axis



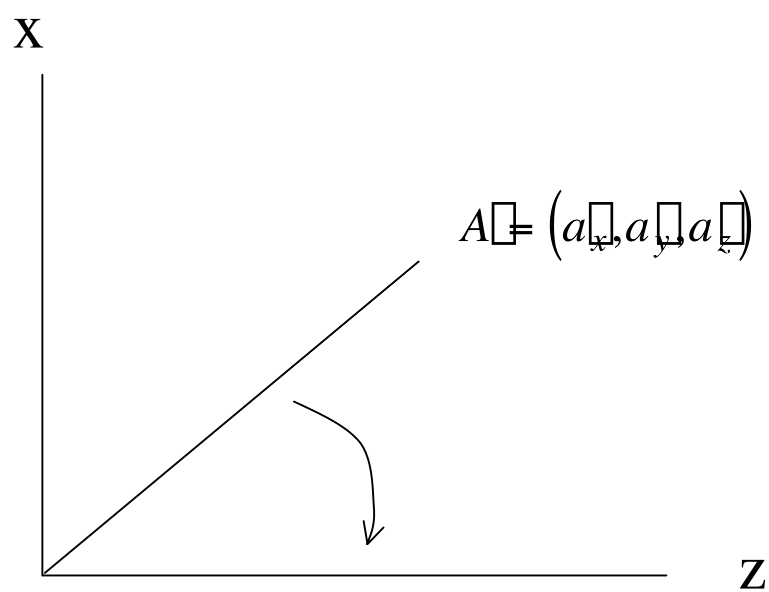
$$d = \sqrt{a_y^{\,2} + a_z^{\,2}}$$

$$\sin \square_x = a_y / d$$

$$\cos \square_x = a_z / d$$

No need to compute angles, just put sines and cosines into rotation matrices

Rotation about an arbitrary axis



Apply $R_x(\theta_x)$ to A and renormalize to get A'

$R_y(\theta_y)$ should be easy, but note that it is clockwise.

Rotation about an arbitrary axis

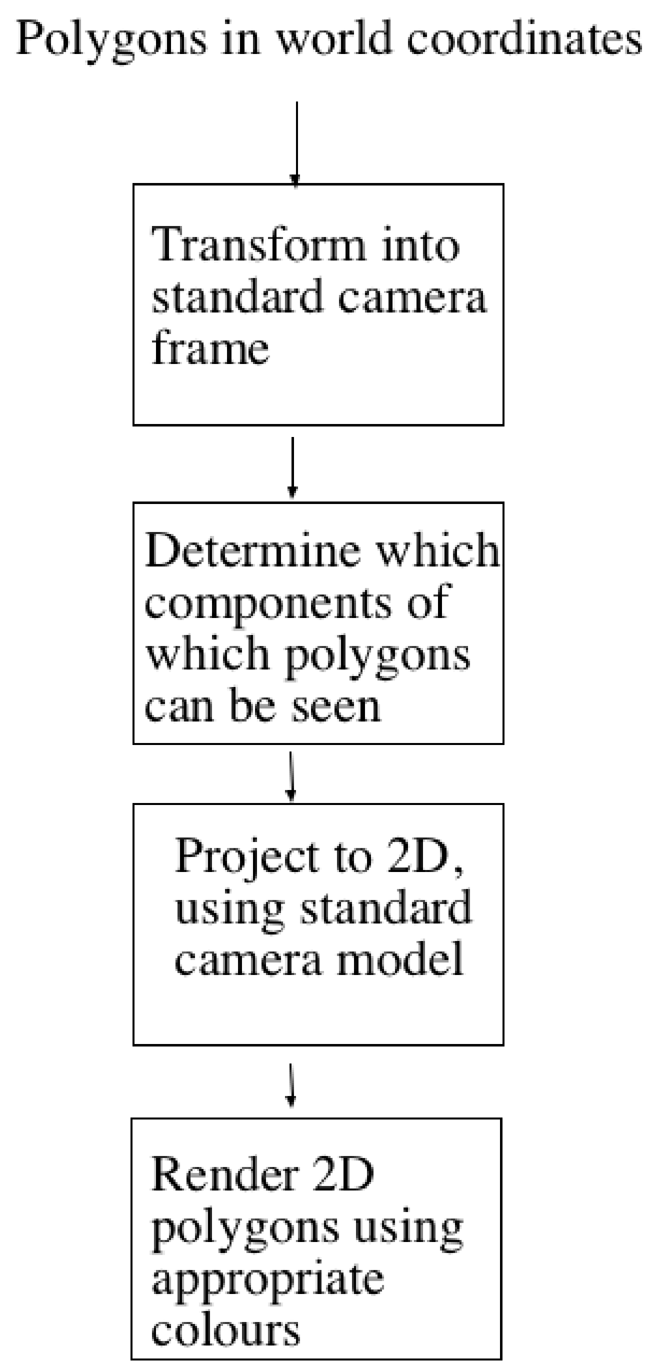
Final form is

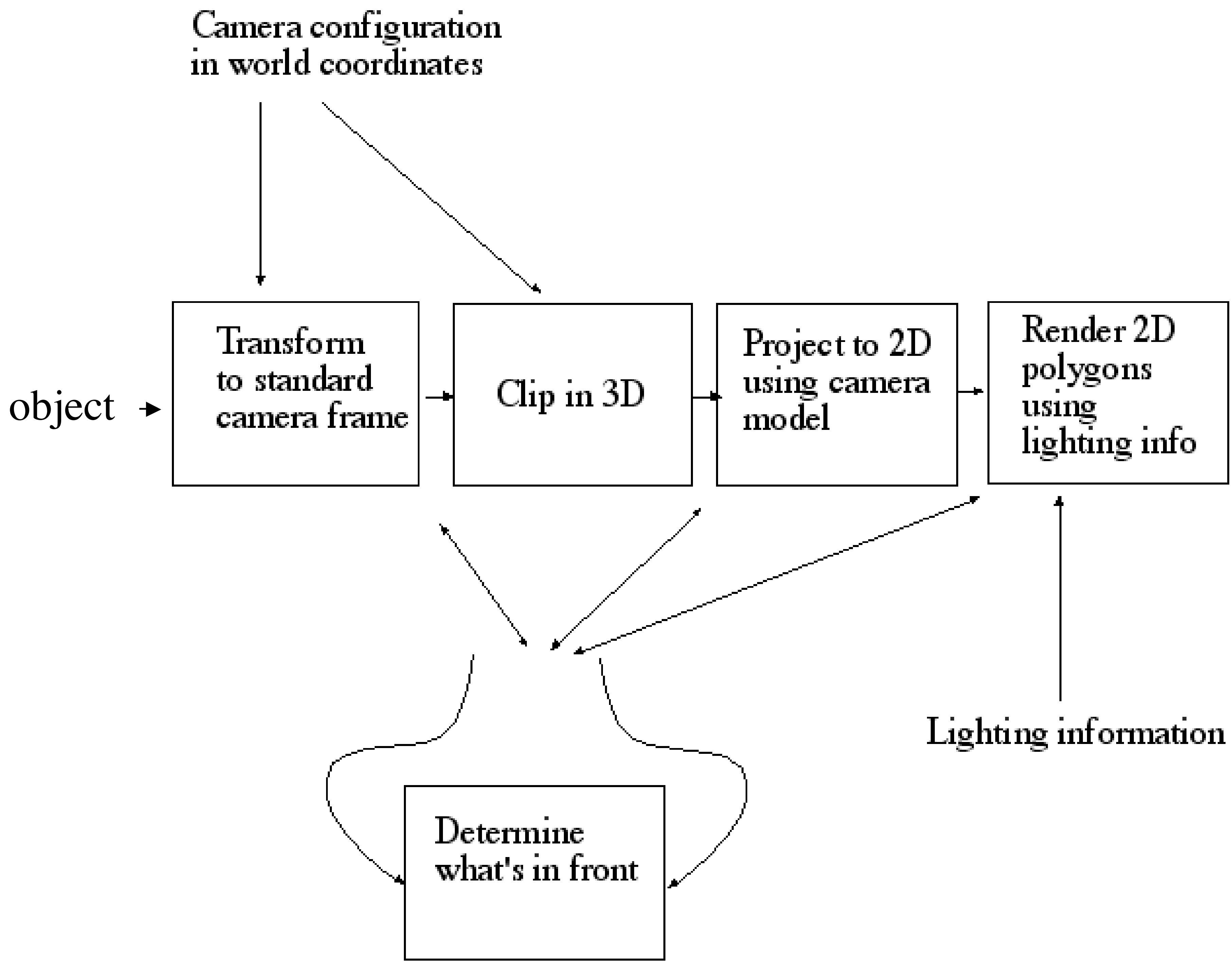
$$R_x(\square\square_x)R_y(\square\square_y)R_z(\square_z)R_y(\square_y)R_x(\square_x)$$

3D Graphics Concepts

(Watt ch. 5, Foley et al ch. 6)

- Modeling: For now, objects will be collections of polygons in 3D. Complex shapes will be many small polygons.
- Issues:
  - Which polygons can be seen? (some polygons hide others, and some are outside the relevant volume of space and need to be clipped).
  - Where do they go in the 2D image? (key abstraction is a virtual camera)
  - How bright should they be? (for example, to make it look as if we are looking at a real surface)

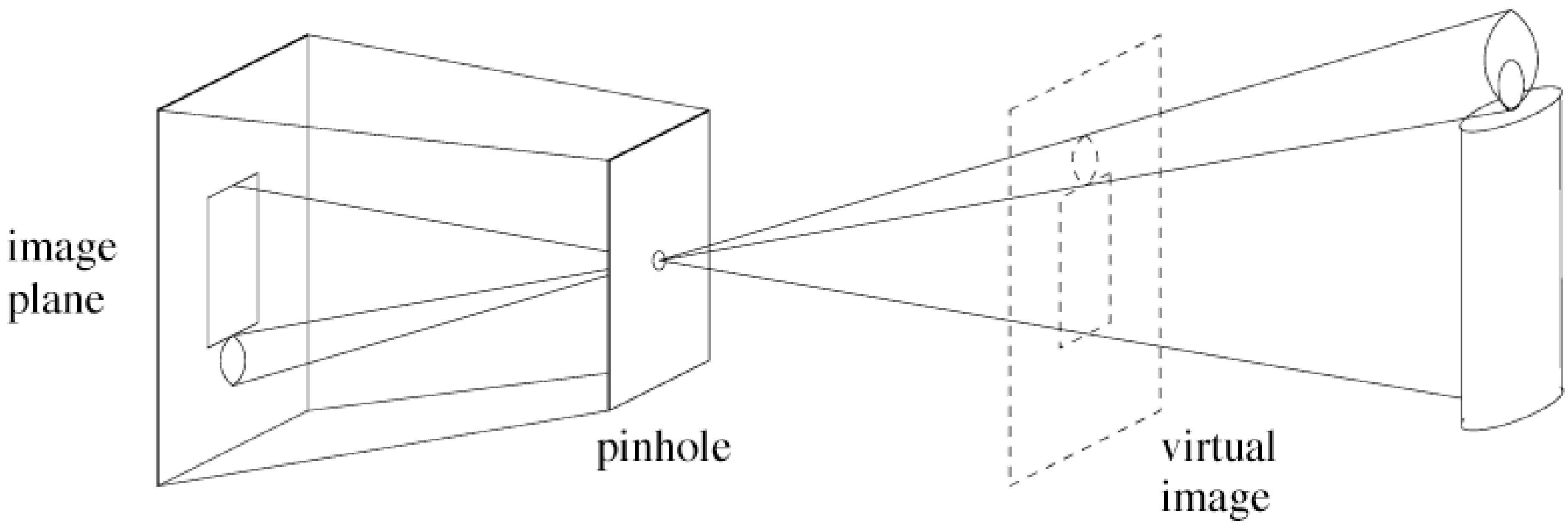




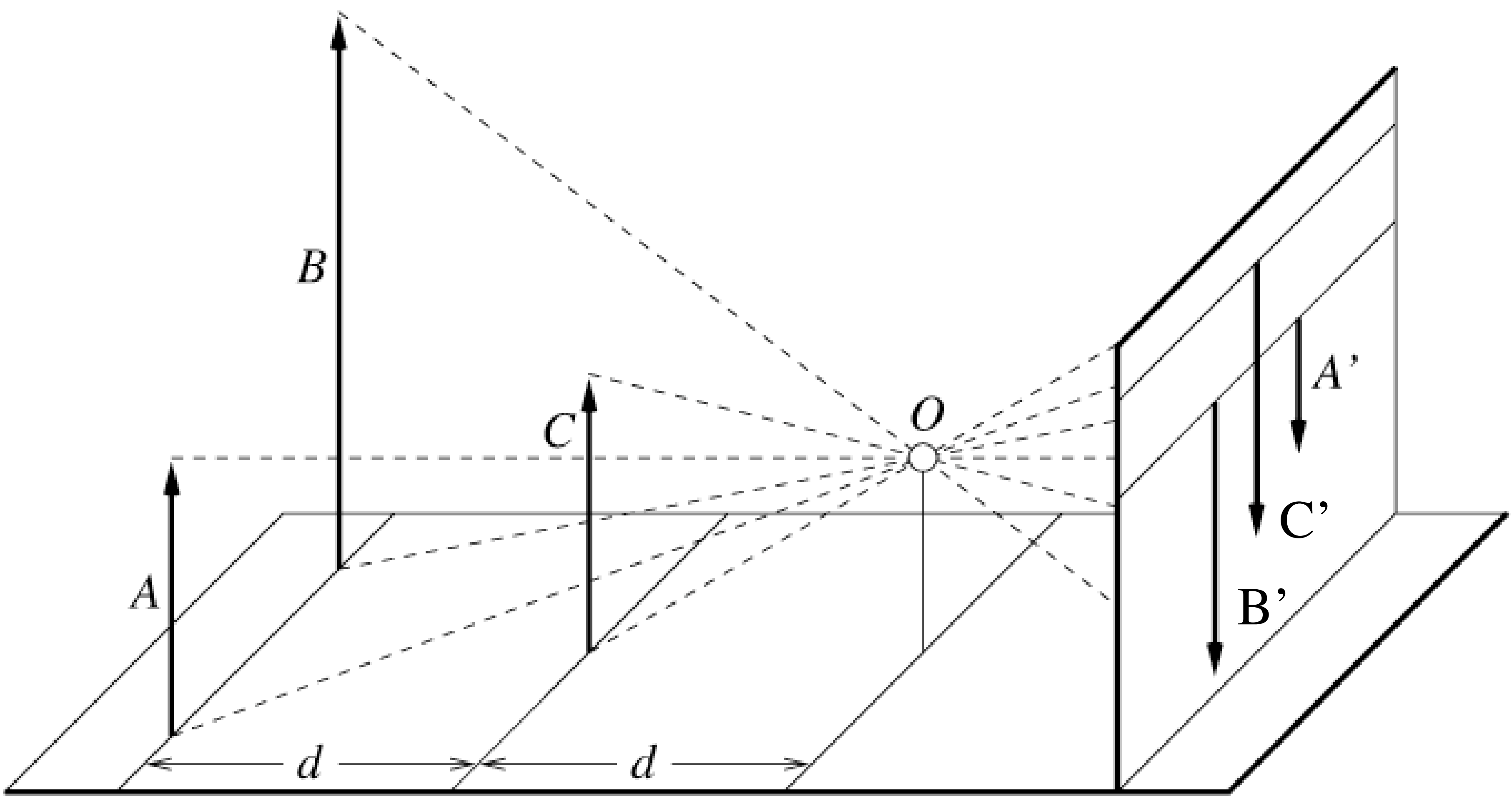
Camera configuration in world coordinates
object
Transform to standard camera frame
Clip in 3D
Project to 2D using camera model
Render 2D polygons using lighting info
Lighting information
Determine what's in front

Pinhole cameras

- Abstract camera model--box with a small hole in it
- Pinhole cameras work for deriving algorithms--a real camera needs a lens

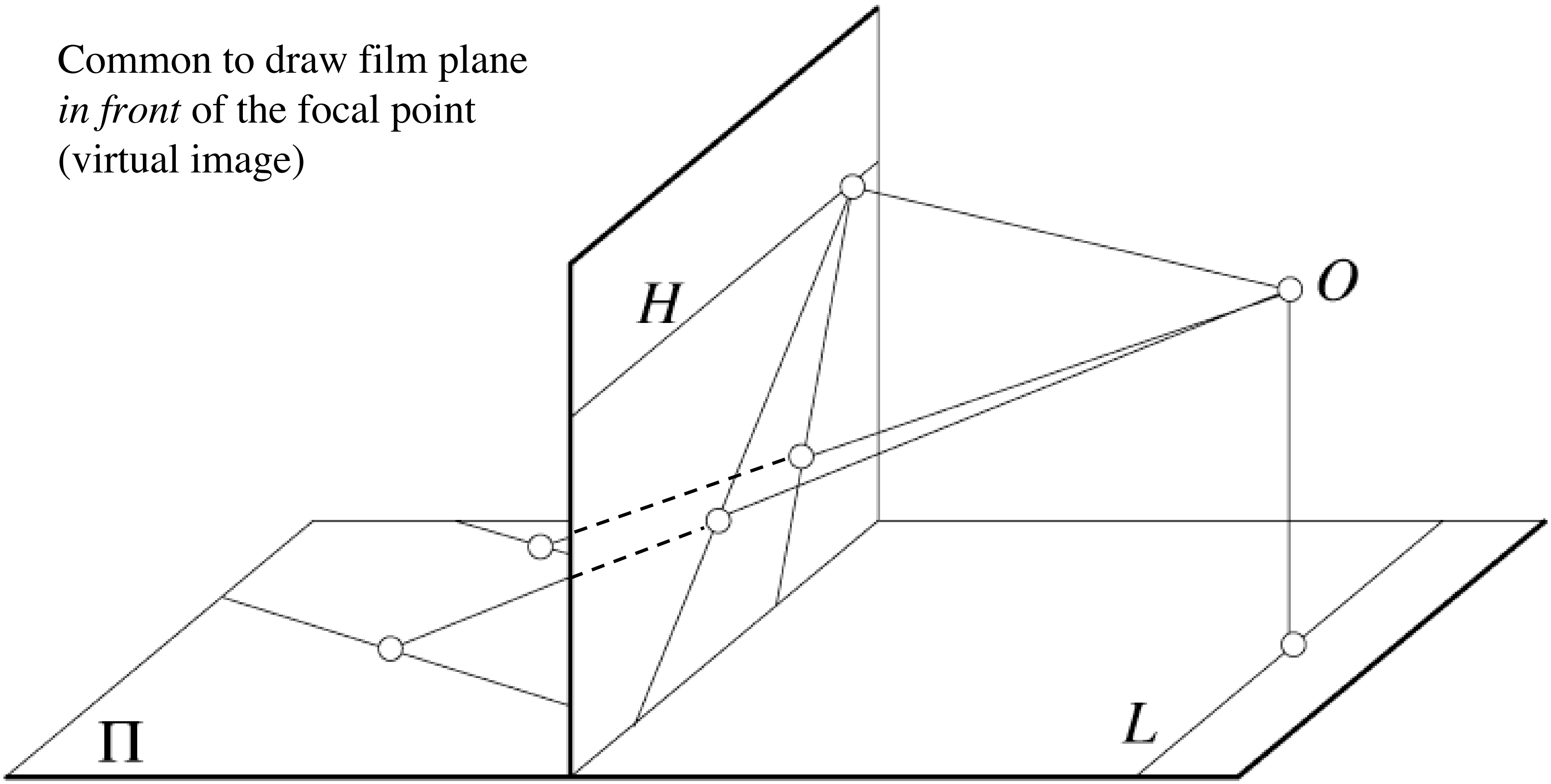


Distant objects are smaller



Parallel lines meet*

Common to draw film plane
in front of the focal point
(virtual image)



*Exceptions?

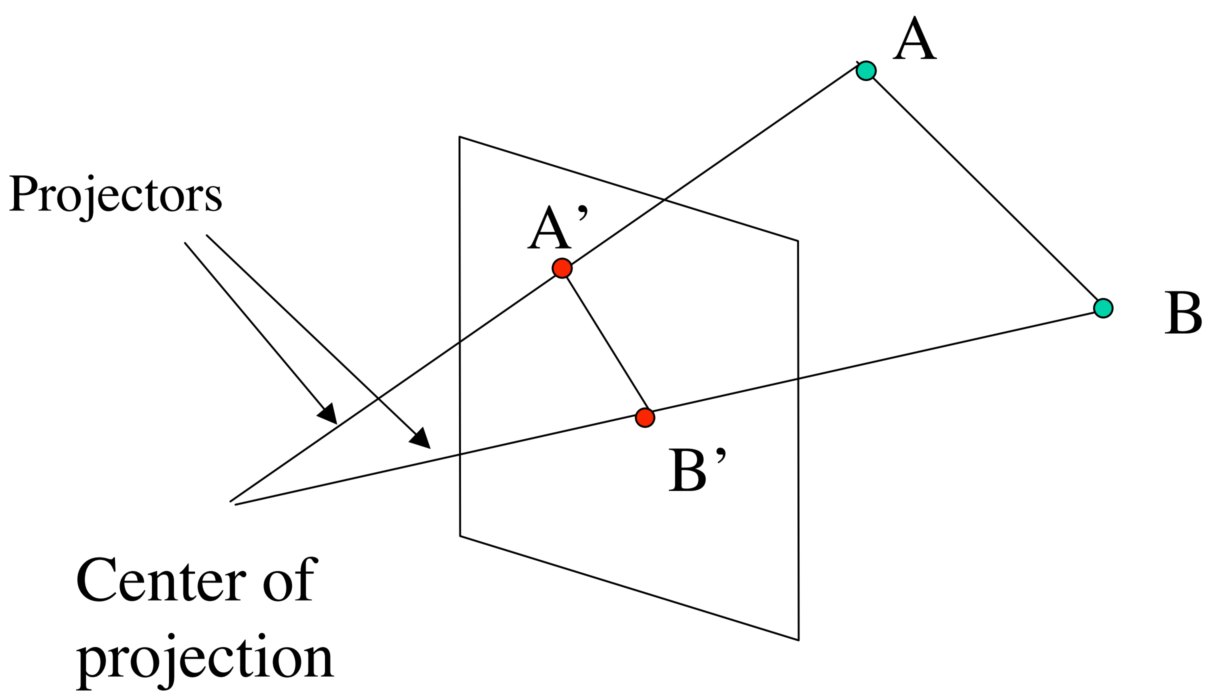
Vanishing points

- Each set of parallel lines (=direction) meets at a different point
  - The *vanishing point* for this direction
- Sets of parallel lines on the same plane lead to *collinear* vanishing points.
  - The line is called the *horizon* for that plane
  - Standard horizon is the horizon of the ground plane.
- One way to spot fake images

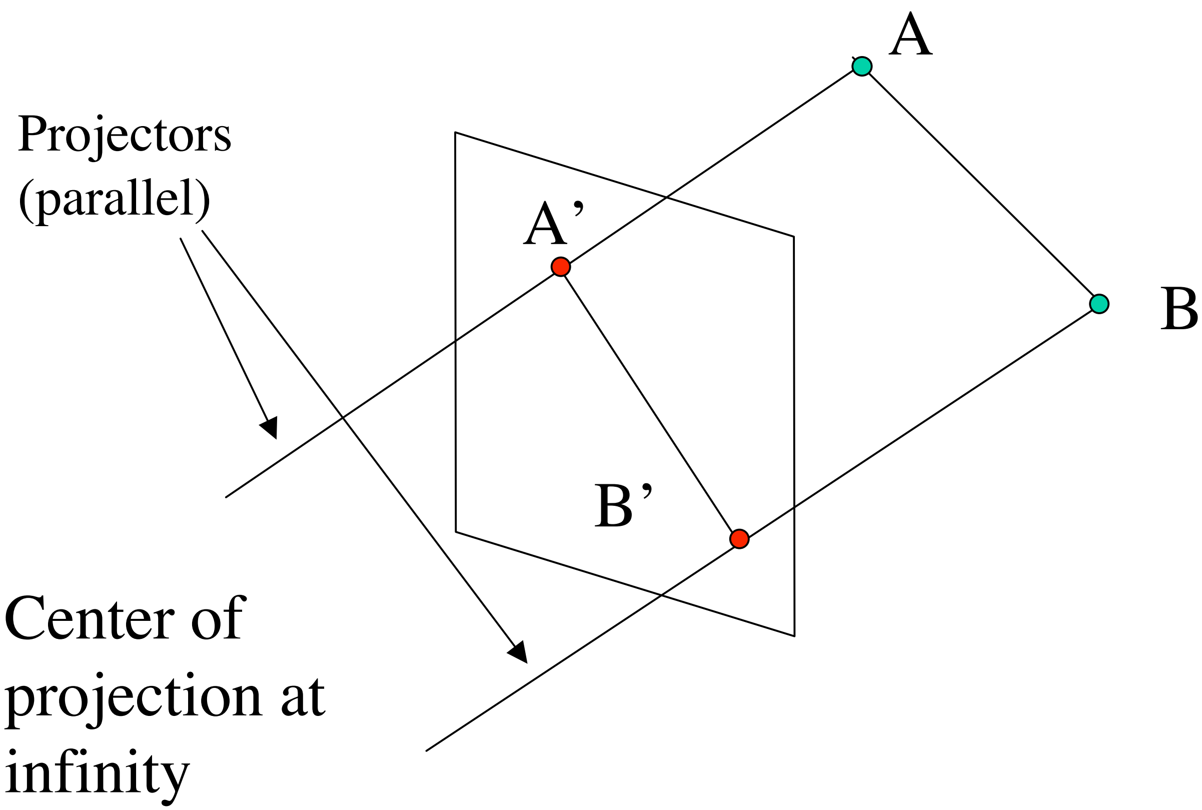
Projections

- Mathematical definition of a projection: PP=P
- (Doing it a second time has no effect).
- Generally rank deficient (non-invertable)--exception is P=I
- Transformation looses information (e.g., depth)
- Given a 2D image, there are many 3D worlds that could have lead to it.

Projections



Parallel Projection



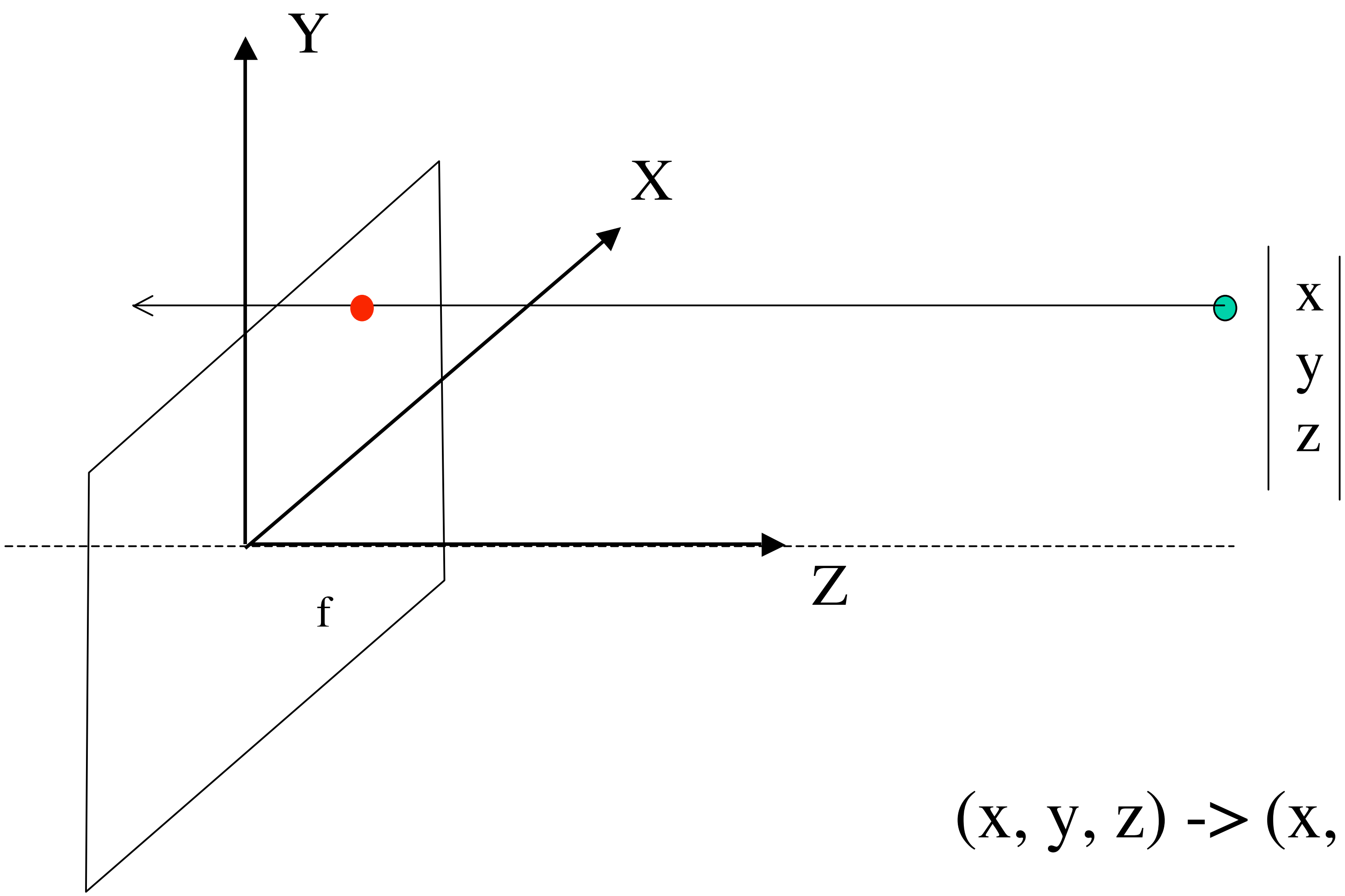
Parallel Projection

Parallel lines remain parallel, some 3D measurements can be made using 2D picture

Does not give realistic 3D view because eye is more like perspective projection.

If projection plane is perpendicular to projectors the projection is orthographic (e.g., top view, side view, front view)

Orthographic example (onto z=0)



(x, y, z) -> (x, y, 0)

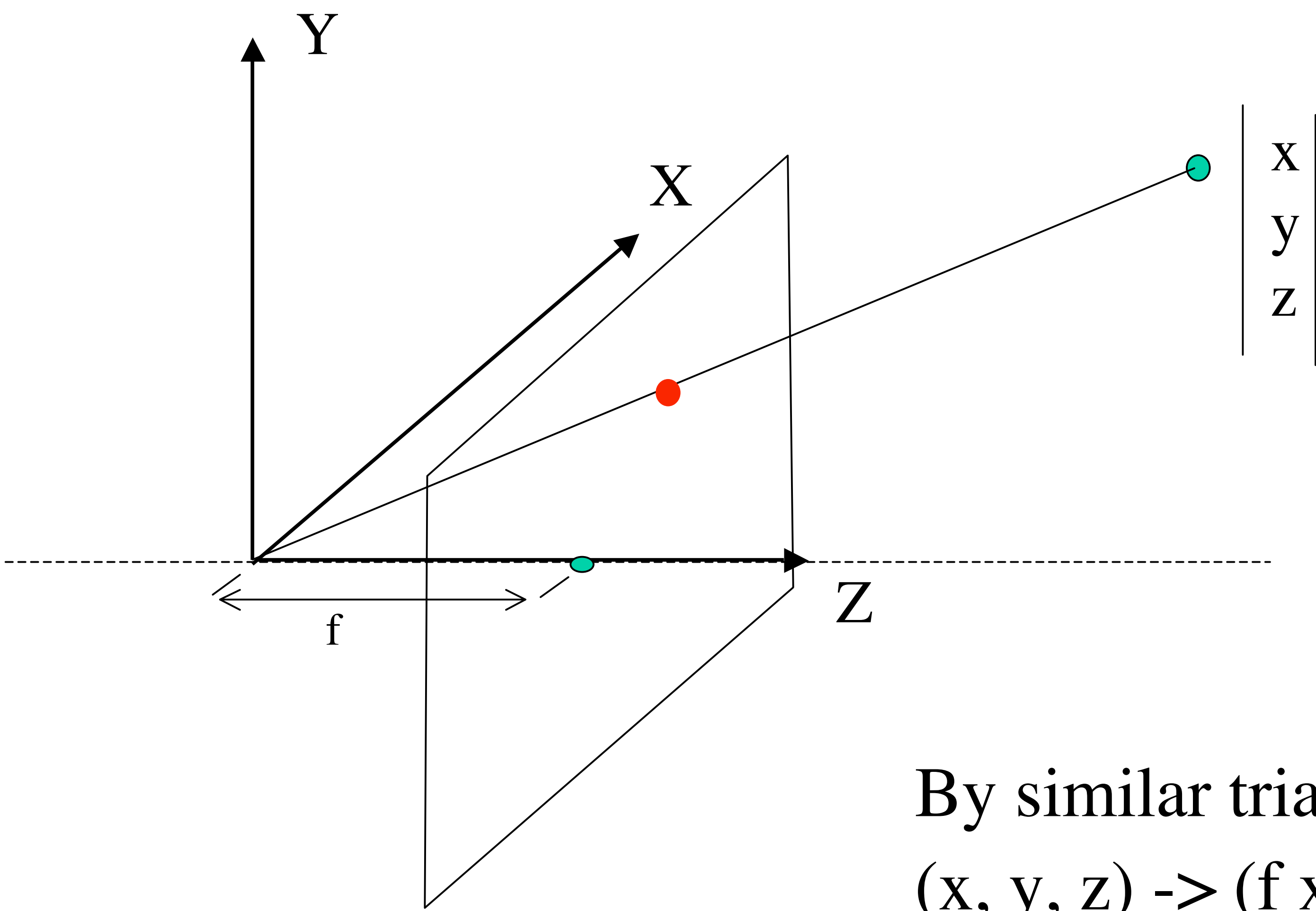
The camera matrix

$$\begin{bmatrix} x \\ y \\ 0 \\ w \end{bmatrix} = \quad ? \quad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

The camera matrix

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Perspective example (onto z=f)



By similar triangles,

$(x, y, z) \rightarrow (f\,x/z,\ f\,y/z,\ f)$

The equation of projection

- In homogeneous coordinates

$$(x,\ y,\ z,\ 1)\ \square\ (f\frac{x}{z},\ f\frac{y}{z},\ f,\ 1)$$

- Equivalently

$$(x,\ y,\ z,\ 1)\ \square\ (x,\ y,\ z,\ \frac{z}{f})$$

- (Now H.C. are being used to store foreshortening)

The camera matrix

$$\begin{bmatrix} x \\ y \\ z \\ \frac{z}{f} \end{bmatrix} = \begin{bmatrix} & & & \\ & & ? & \\ & & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

The camera matrix

$$\begin{bmatrix} x \\ y \\ z \\ \frac{z}{f} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & \frac{1}{f} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$