B-splines - I

- Now consider stitching together curves which do not necessarily pass through the control points.
- Local control
- Blending functions are non-zero over limited range--thus they are like "switches"
- In the simplest case of uniformly spaced control points, the blending functions will be shifted versions of the same function.

B-splines - II

- Curve (general case):

$$X(t) = \sum_{k=0}^{n} P_k B_{k,d}(t)$$

- The “degree parameter” d is:

$$2 \le d \le n+1$$

- Note the “degree” of the polynomial is d-1 (not d).

- We have n+1 control points, and n+1 blending functions. The most common case is d=4.

B-Spline Blending Functions

- Knots
  - parameter values where curve segments meet
- There are n+d+1 knots for B-spline

  $$\left(t_0, t_1, \ldots, t_{n+d}\right)$$

  where $t_0 \le t_1 \le \ldots \le t_{n+d}$
- B-spline is defined for the range $\left(t_{d-1}, \ldots, t_{n+1}\right)$
  - So all curve points have contributions from d control points
  - All sections of curves behave the same (later we will see how to interpolate the endpoints.
- Blending functions for degree parameter d have support (are non-zero) for d subintervals.

B-Spline Blending Functions

- Blending functions

$$B_{k,1}(t) = \begin{cases} 1 & t_k \le t < t_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k} \right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}} \right) B_{k+1,d-1}(t)$$

- If knots are repeated we use 0/0=0

B-Spline Blending Functions

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k}\right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}}\right) B_{k+1,d-1}(t)$$

$$B_{k,1}(t) = \begin{cases} 1 & t_k \le t < t_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

So, assuming uniformly spaced knots,

$$B_{k,2}(t) = \quad \textbf{?}$$

B-Spline Blending Functions

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$$B_{k,1}(t) = \begin{cases} 1 & t_k \le t < t_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

So, assuming uniformly spaced knots,

$$B_{k,2}(t) = \begin{cases} \left(\dfrac{t - t_k}{\Delta t}\right) & t_k \le t < t_{k+1} \\ \left(\dfrac{t_k - t}{\Delta t}\right) & t_{k+1} \le t < t_{k+2} \\ 0 & \text{otherwise} \end{cases}$$

B-Spline Blending Functions

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k} \right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}} \right) B_{k+1,d-1}(t)$$

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So, assuming uniformly spaced knots,

$$B_{k,3}(t) = \ \ ?$$

B-Spline Blending Functions

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k}\right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}}\right) B_{k+1,d-1}(t)$$

$$B_{k,2}(t) = \begin{cases} \left(\frac{t - t_k}{\Delta t}\right) & t_k \le t < t_{k+1} \\ \left(\frac{t_k - t}{\Delta t}\right) & t_{k+1} \le t < t_{k+2} \\ 0 & \text{otherwise} \end{cases}$$

So, assuming uniformly spaced knots,

$$B_{k,3}(t) = \begin{cases} \text{quadratic function} & t_k \le t < t_{k+1} \\ \text{quadratic function} & t_{k+1} \le t < t_{k+2} \\ \text{quadratic function} & t_{k+2} \le t < t_{k+3} \\ 0 & \text{otherwise} \end{cases}$$

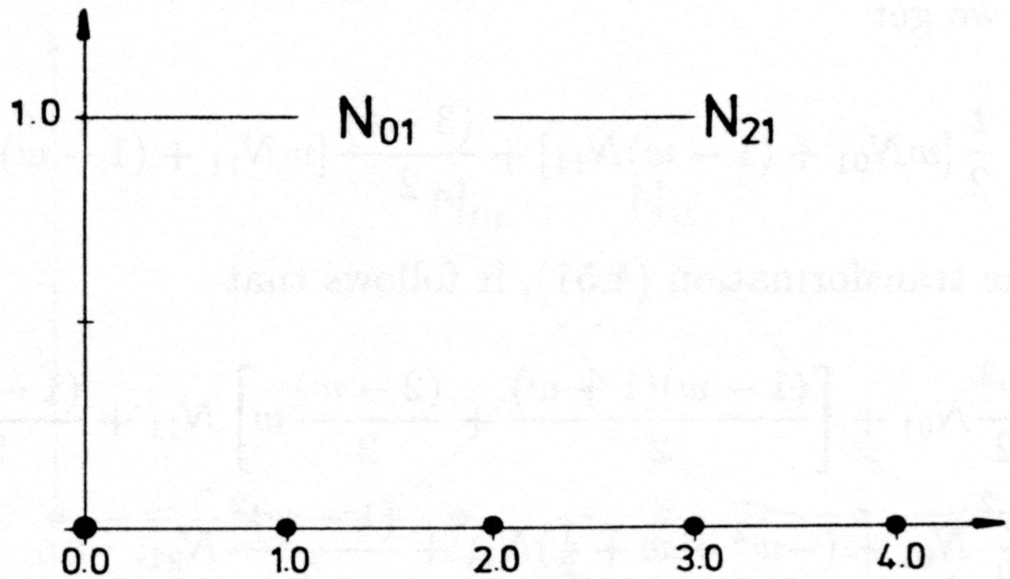


Fig. 4.22c. The B-splines N_{01}, N_{21}.

These figures show blending functions with a uniform knot vector, knots at 0, 1, 2, etc. Note that N is the same as our B

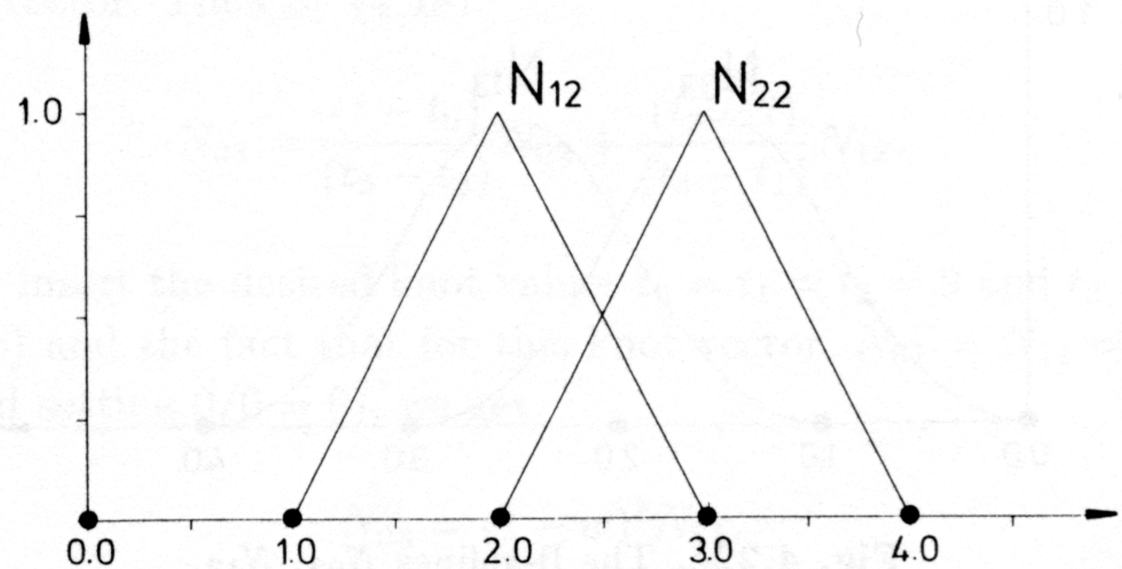


Fig. 4.22d. The B-splines N_{12}, N_{22}.

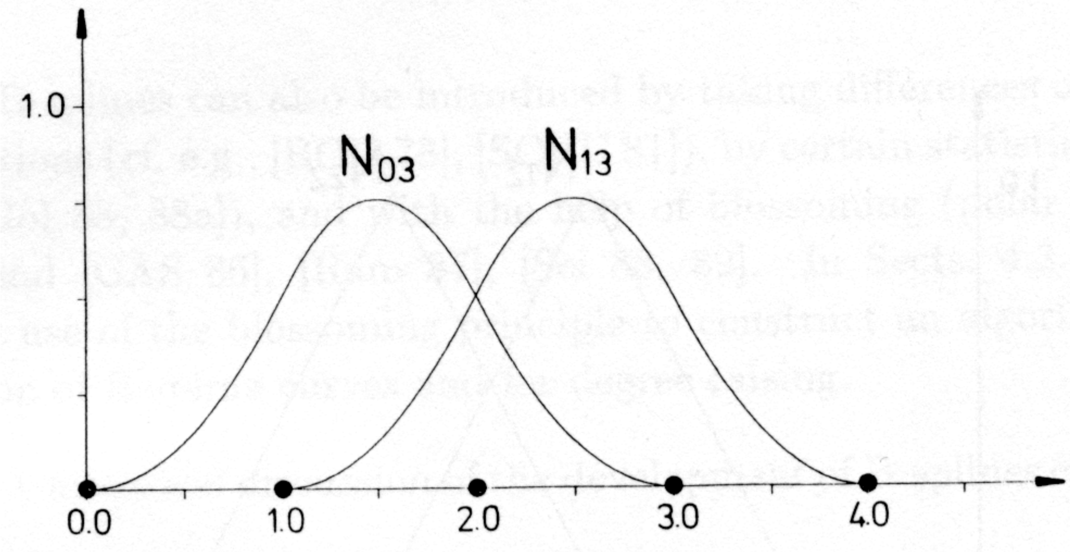
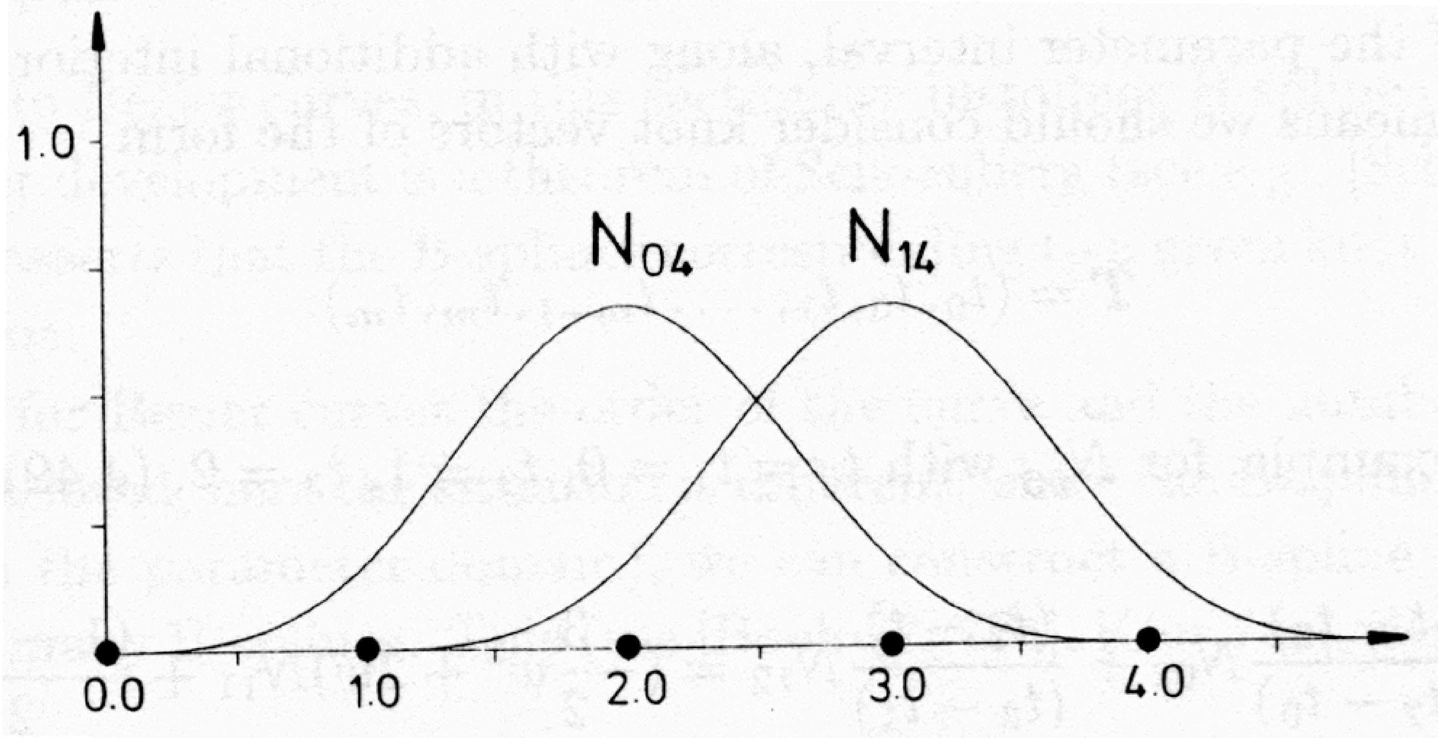


Fig. 4.22e. The B-splines N_{03}, N_{13}.



Matrix form of Uniform Cubic B-Spline Blending Functions

$$M_B = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

Closed B-Splines

- Periodically extend the control points and the knots

$$P_{n+1} = P_0$$

$$t_{n+1} = t_0$$

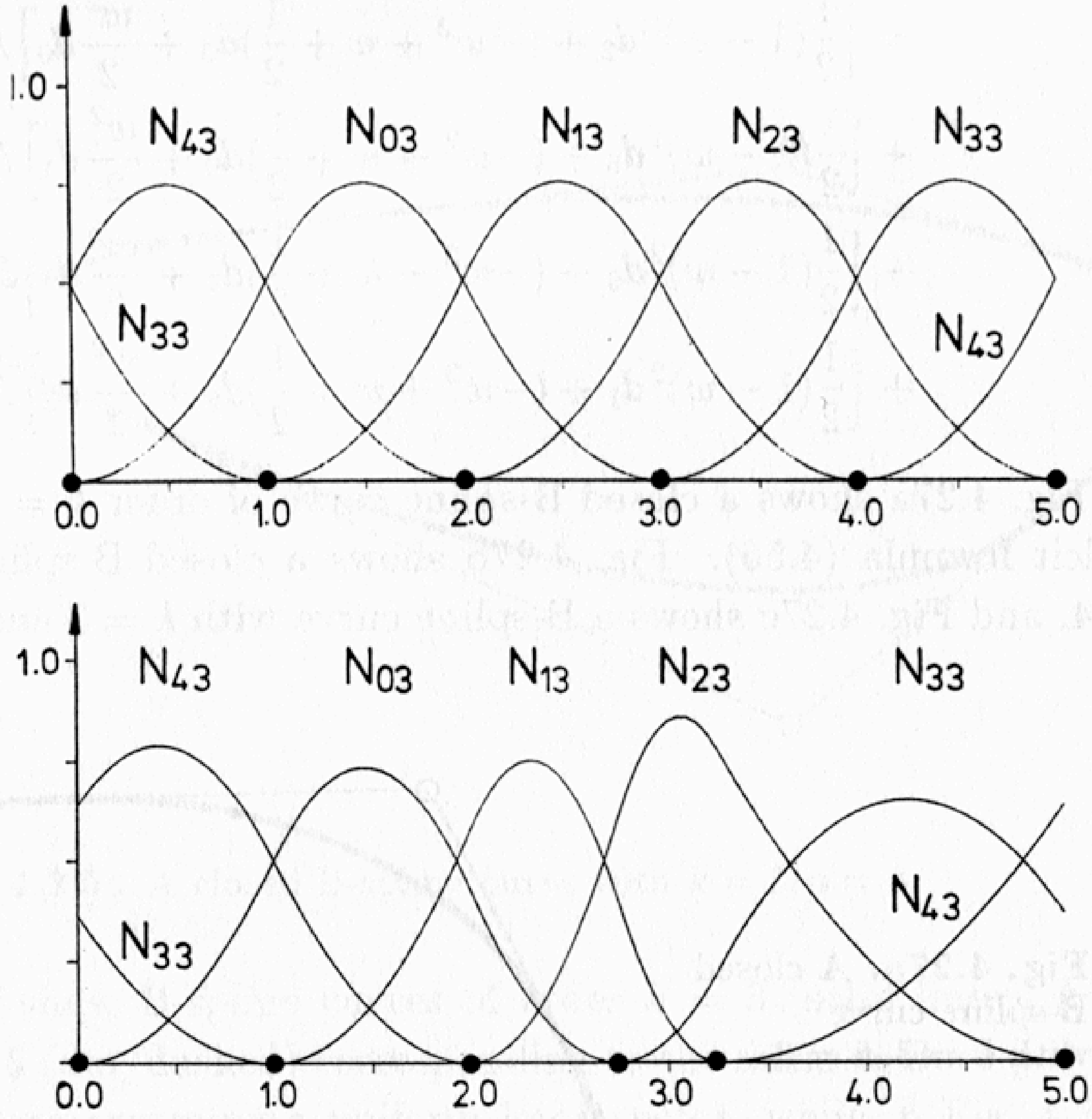


Fig. 4.26a.

Fig. 4.26b.

Fig. 4.26. B-splines with uniform and non-uniform knot vectors for a closed B-spline curve.

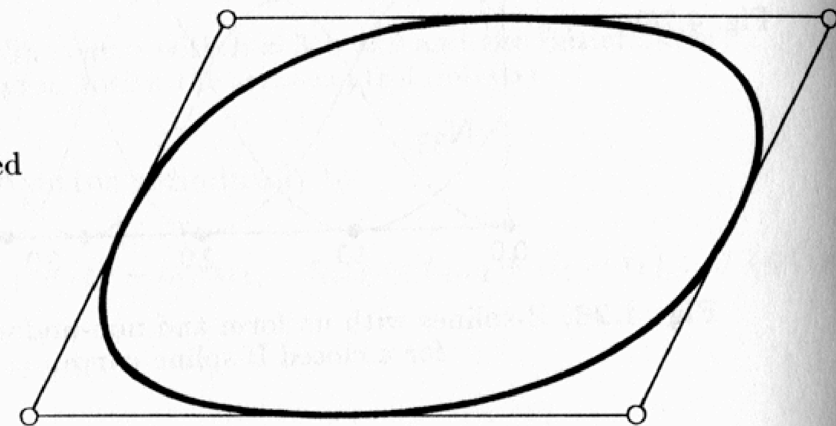


Fig. 4.27a. A closed B-spline curve with $k = 3, n = 3$.

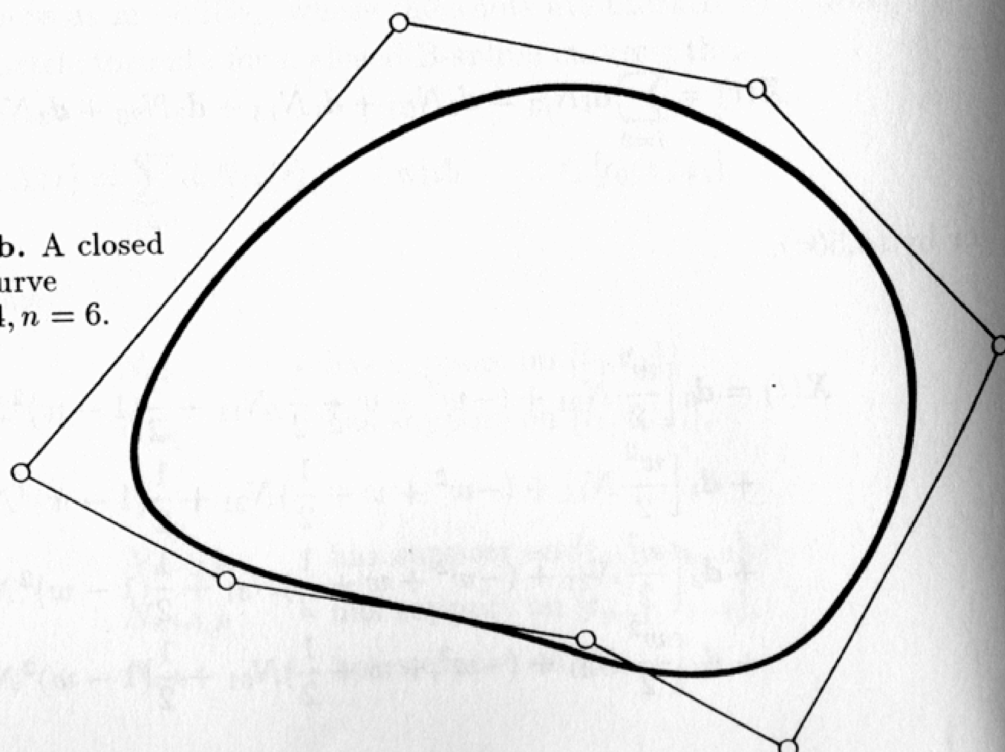


Fig. 4.27b. A closed B-spline curve with $k = 4, n = 6$.

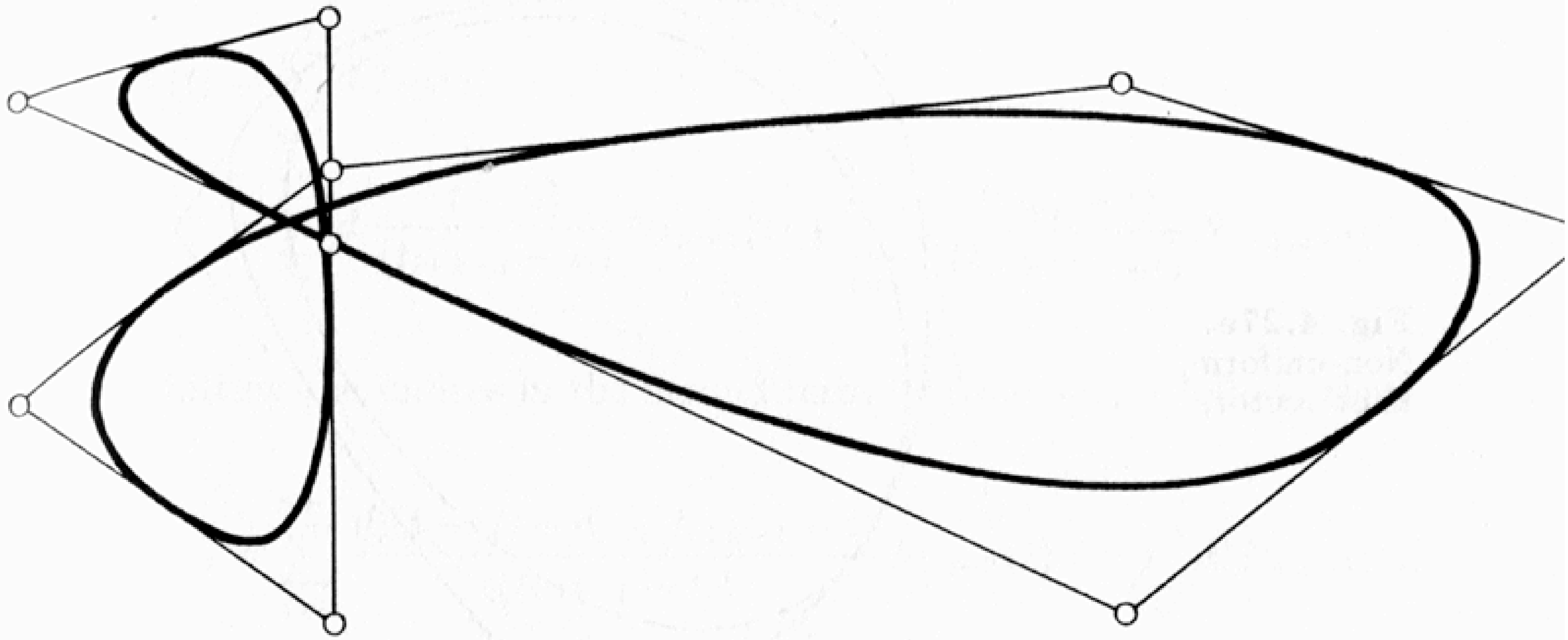
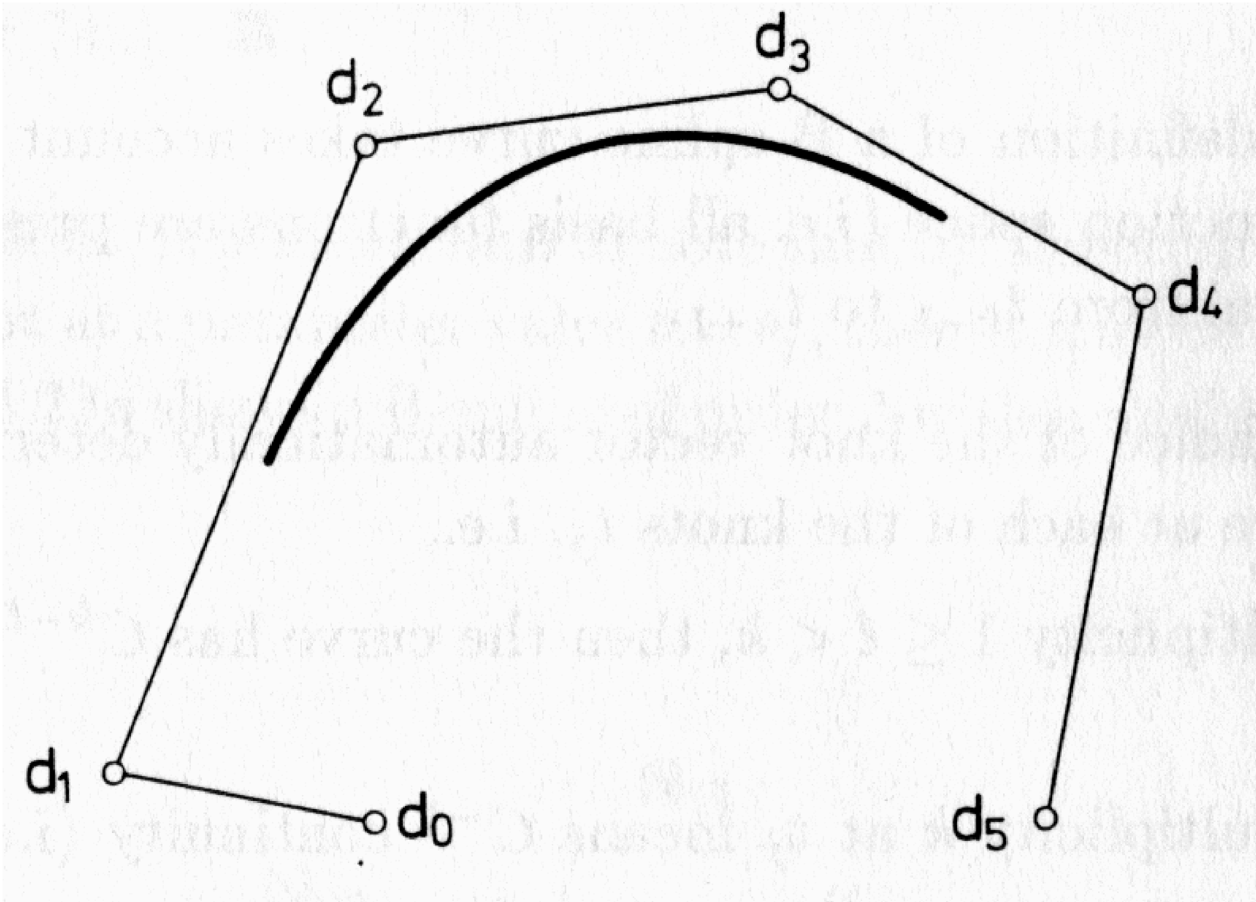


Fig. 4.27c. A closed B-spline curve with $k = 3, n = 8$.



Recall that each curve section is a blend of d control points.

What if we want to interpolate the endpoints?

Repeated knots

- Definition works for repeated knots (if we are understanding about 0/0)
- Repeated knot reduces continuity. A B-spline blending function has continuity C^{d-2}; if the knot is repeated m times, continuity is now C^{d-m-1}
- e.g. -> quadratic B-spline (i.e. order 3) with a double knot

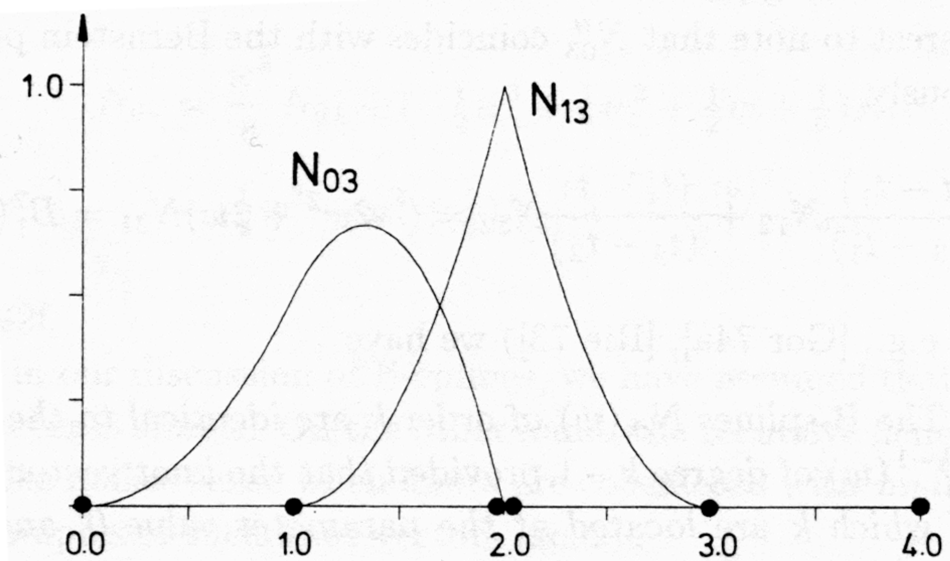


Fig. 4.22g. A quadratic B-spline with a double knot.

Most useful case

- Select the first d and the last d knots to be the same
  - we then get the first and last points lying on the curve
  - also, the curve is tangent to the first and last segment
- E.g. cubic case below
- Notice that a control point influences at most d parameter intervals - **local control**

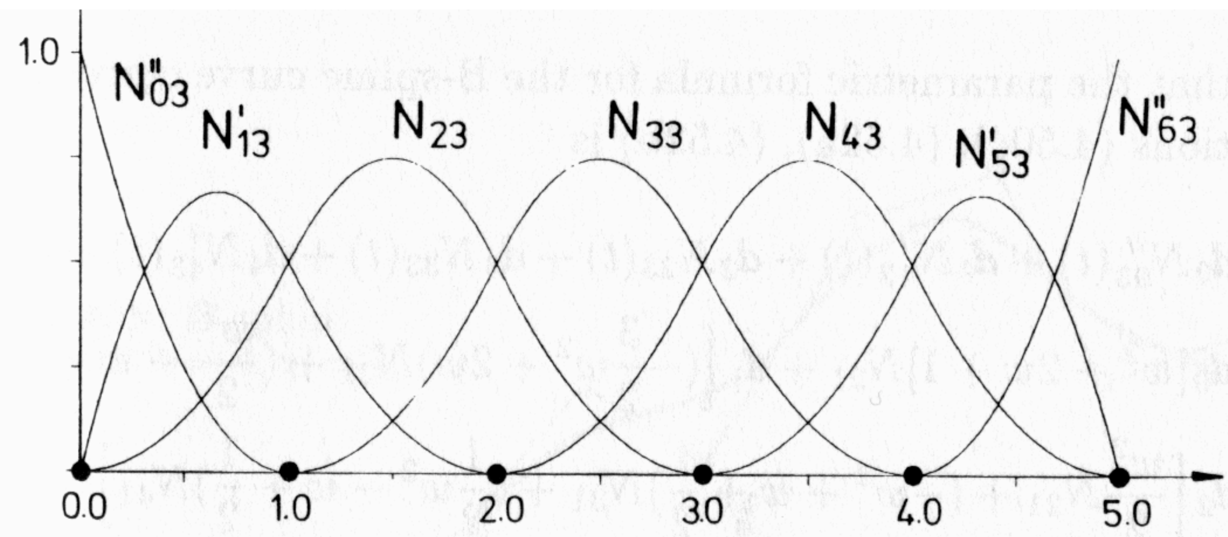


Fig. 4.24a. B-splines for an open B-spline curve with uniform knot vector.

Fig. 4.25a. B-spline curve with $k = 3$, $n = 5$.

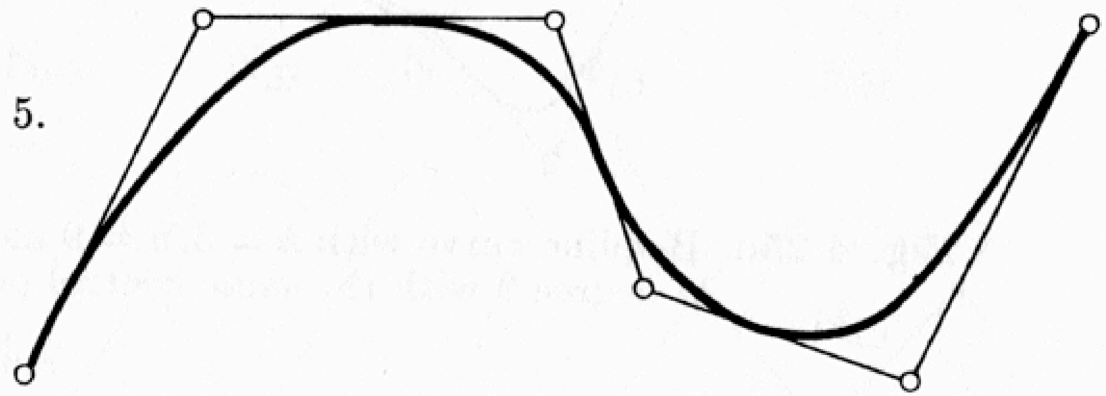
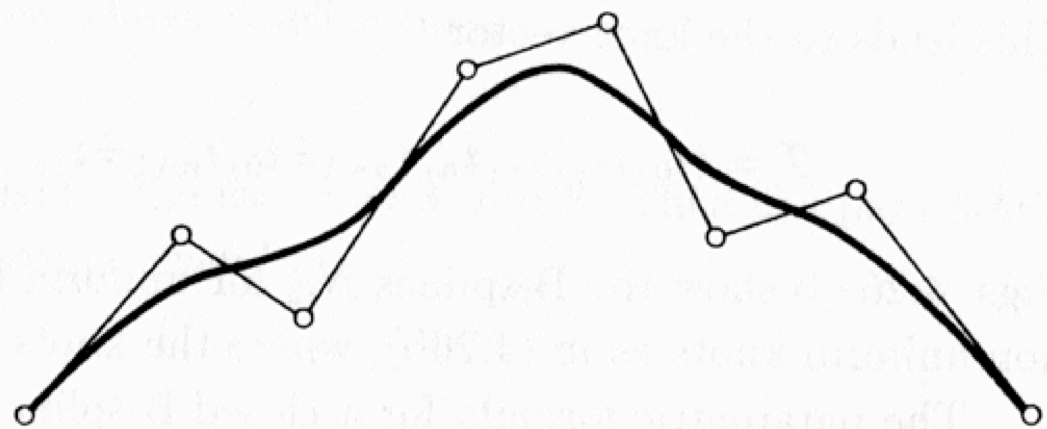


Fig. 4.25b. B-spline curve with $k = 4$, $n = 7$.



k is our d - top curve has order 3, bottom order 4

Example of blending function with repeated knots at the endpoints and non-uniform spacing of interior knots

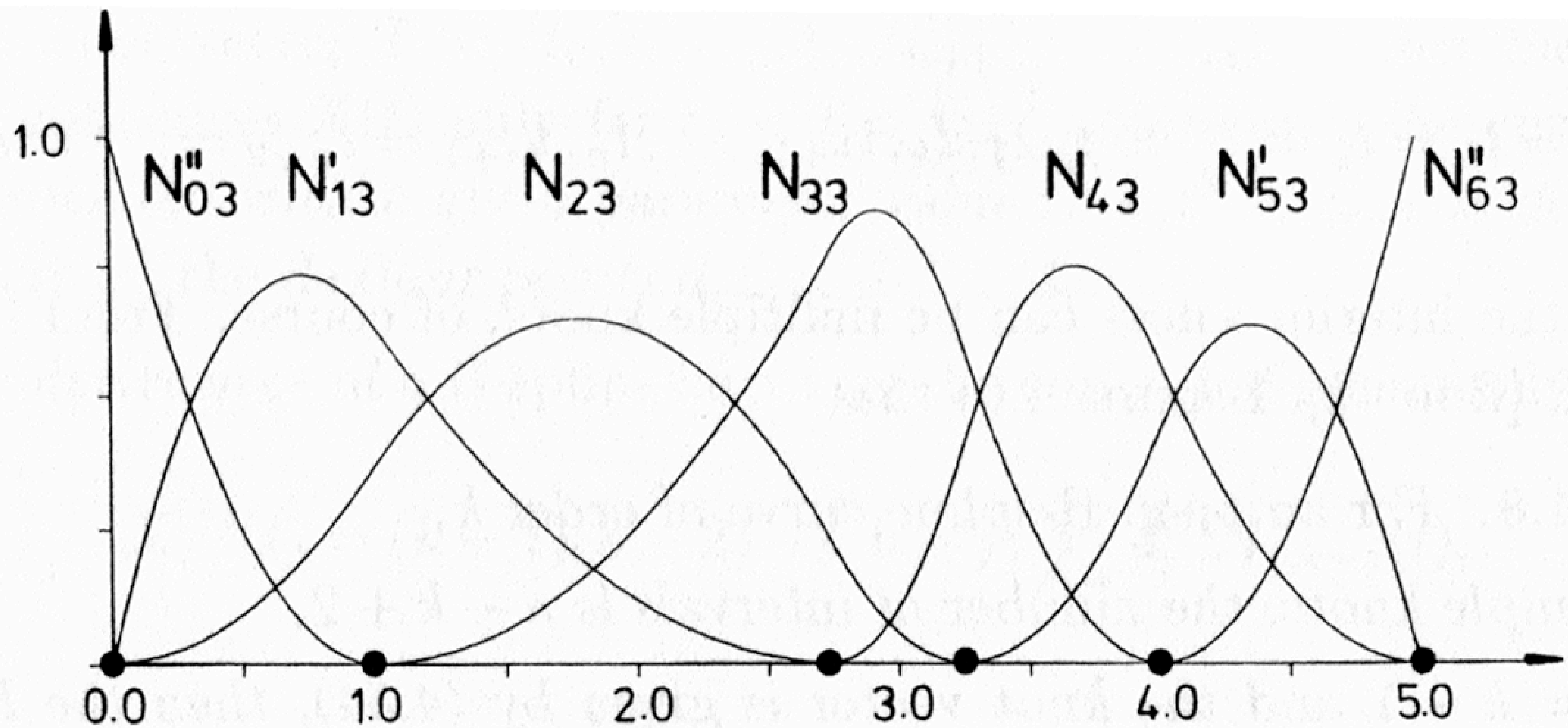


Fig. 4.24b. B-splines for an open B-spline curve with non-uniform knot vector.

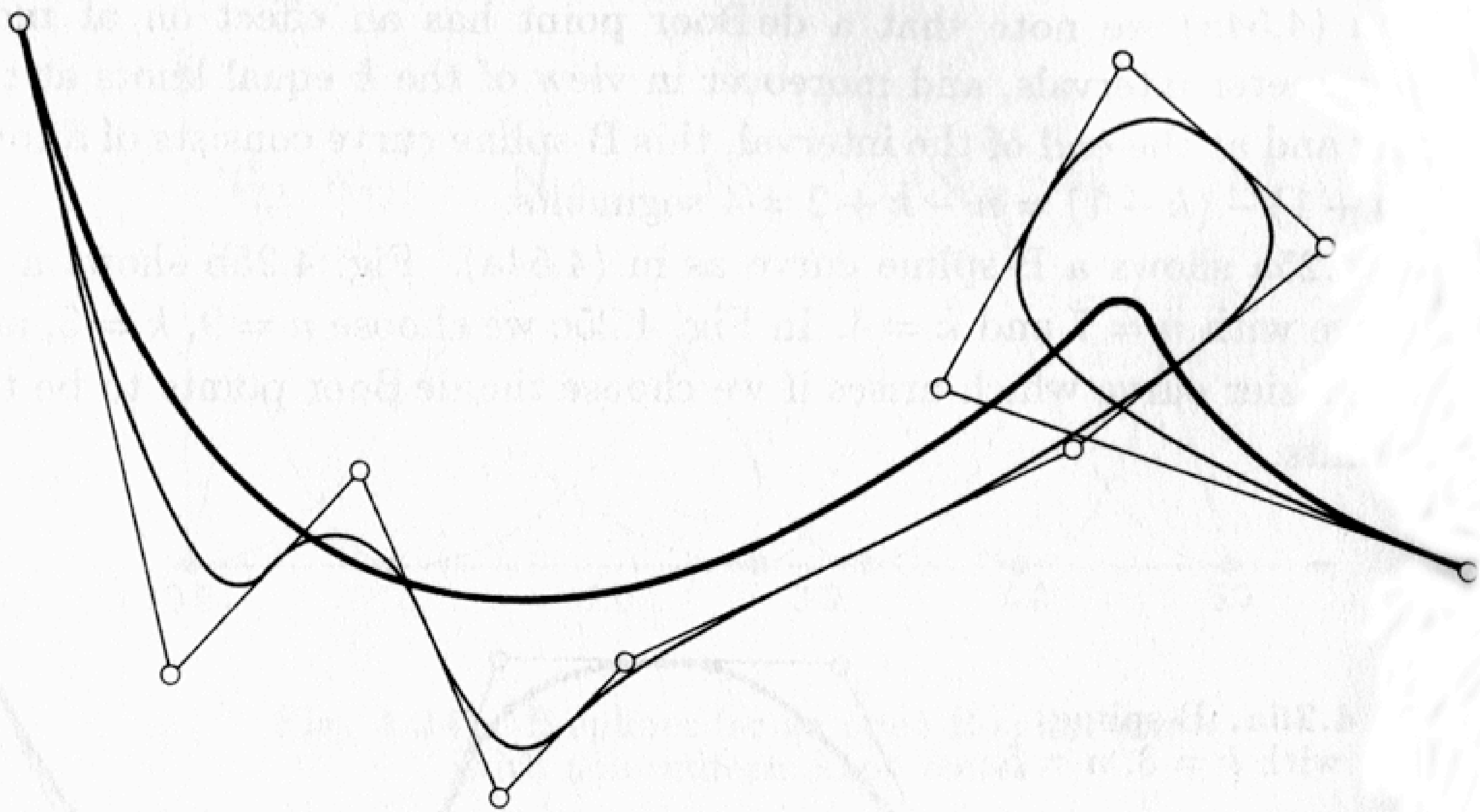


Fig. 4.25c. B-spline curve with $k = 3, n = 9$ and the Bézier curve of degree 9 with the same control polygon.

Bézier curve is the heavy curve

B-Spline properties

- For a B-spline curve of order d
  - if m knots coincide, the curve is C^{d-m-1} at the corresponding point
  - if d-1 consecutive* points of the control polygon are collinear, then the curve is tangent to the polygon
  - if d consecutive* points of the control polygon are collinear, then the curve and the polygon have a common segment
  - if d-1 points coincide, then the curve interpolates the common point and the two adjacent sides of the polygon are tangent to the curve
  - each segment of the curve lies in the convex hull of the associated d points

*The fish shaped curve a few slides back have 4 collinear points (d=3), but they are not consecutive so the condition does not hold.