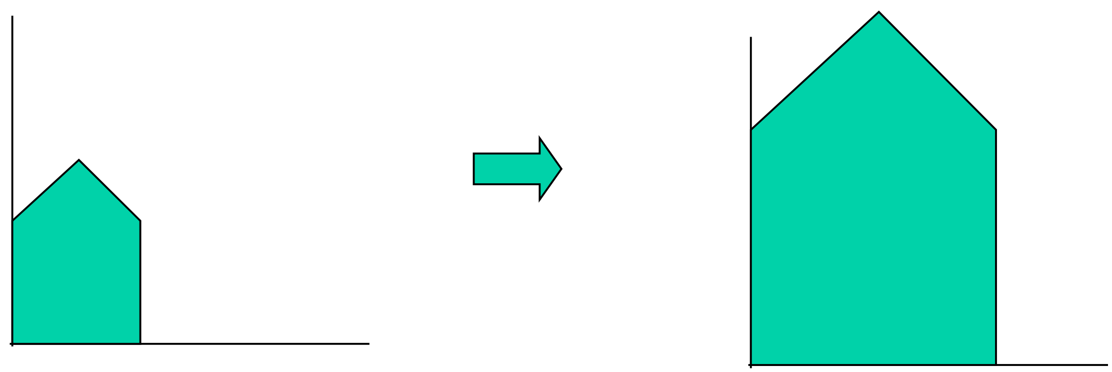
2D Transformations

- Represent transformations by matrices
- To transform a point, represented by a vector, multiply the vector by the appropriate matrix.
- To transform line segments, transform endpoints
- To transform polygons, transform vertices

2D Transformations

- Scale (stretch) by a factor of k

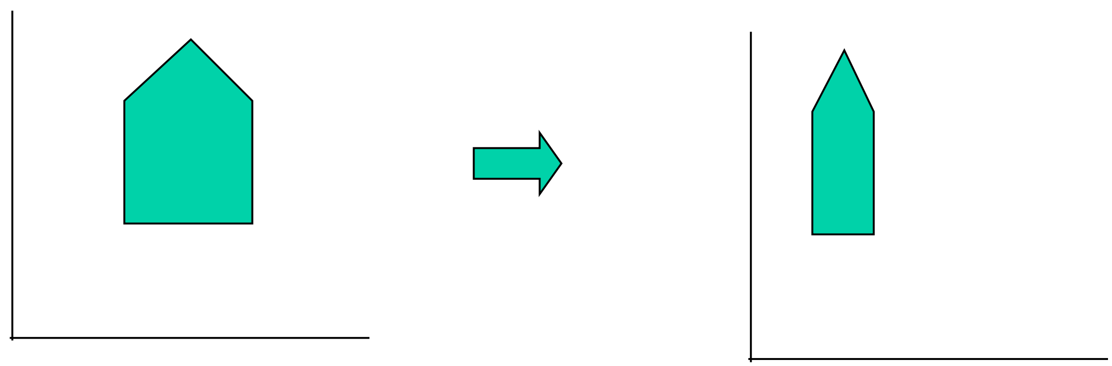


$$M = \begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix}$$

(k = 2 in the example)

2D Transformations

- Scale by a factor of (S_x, S_y)

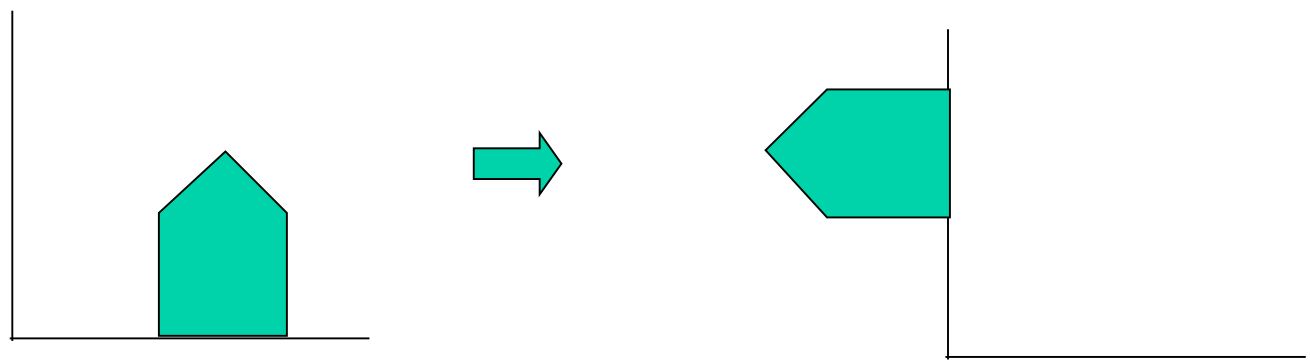


$$M = \begin{vmatrix} S_x & 0 \\ 0 & S_y \end{vmatrix}$$

(Above, $S_x = 1/2$, $S_y = 1$)

2D Transformations

- Rotate around origin by θ (Orthogonal)

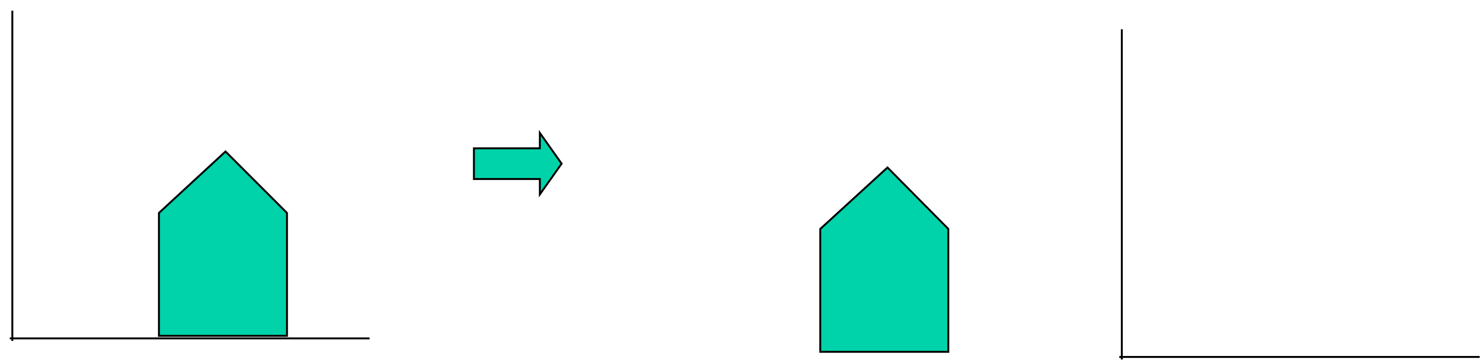


$$M = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

(Above, $\theta = 90^{o}$)

2D Transformations

- Flip over y axis (Orthogonal)

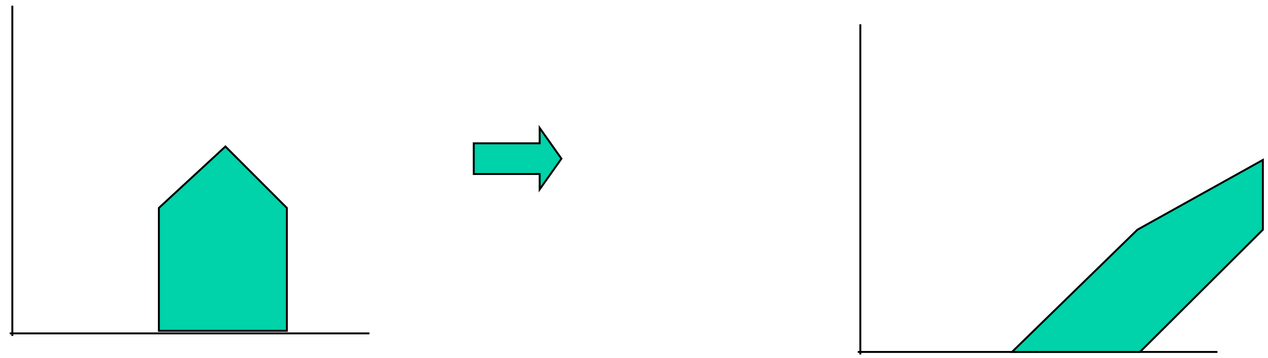


$$M = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$$

Flip over x axis is ?

2D Transformations

- Shear along x axis

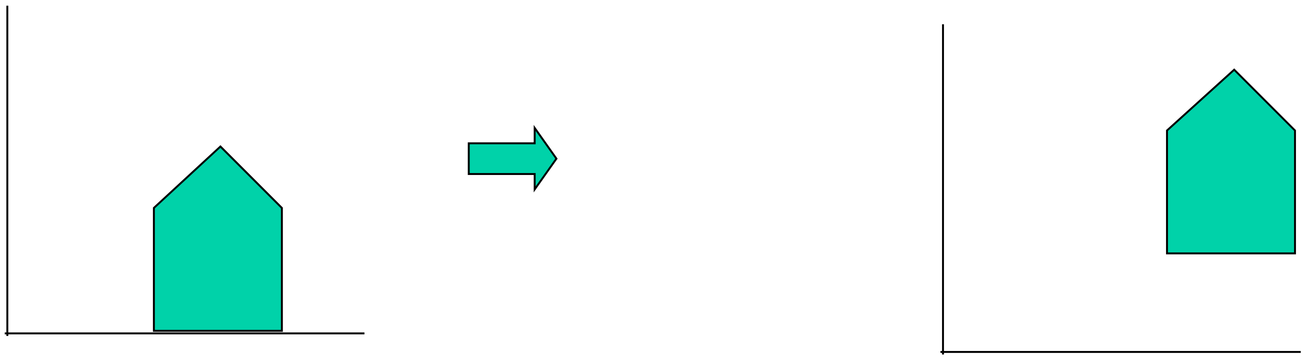


$$M = \begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix}$$

Shear along y axis is ?

2D Transformations

- Translation $(\mathbf{P}_{new} = \mathbf{P} + \mathbf{T})$



$$M = ?$$

Homogenous Coordinates

- Represent 2D points by 3D vectors
- (x,y)-->(x,y,1)
- Now a multitude of 3D points (x,y,W) represent the same 2D point, (x/W, y/W, 1)
- Represent 2D transforms with 3 by 3 matrices
- Can now do translations
- Homogenous coordinates have other uses/advantages (later)

2D Translation in H.C.

$\mathbf{P}_{\text{new}} = \mathbf{P} + \mathbf{T}$

$(x', y') = (x, y) + (t_x, t_y)$

$$M = \begin{vmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{vmatrix}$$

2D Scale in H.C.

$$M = \begin{vmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

2D Rotation in H.C.

$$M = \begin{vmatrix} \cos \square & -\sin \square & 0 \\ \sin \square & \cos \square & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Composition of Transformations (§5.4)

- If we use one matrix, M_1 for one transform and another matrix, M_2 for a second transform, then the matrix for the first transform followed by the second transform is simply $M_2 M_1$
- This generalizes to any number of transforms
- Computing the combined matrix first can save lots of computation

Composition Example

- Matrix for rotation about a point, P
- Problem--we only know how to rotate about the origin.

Composition Example

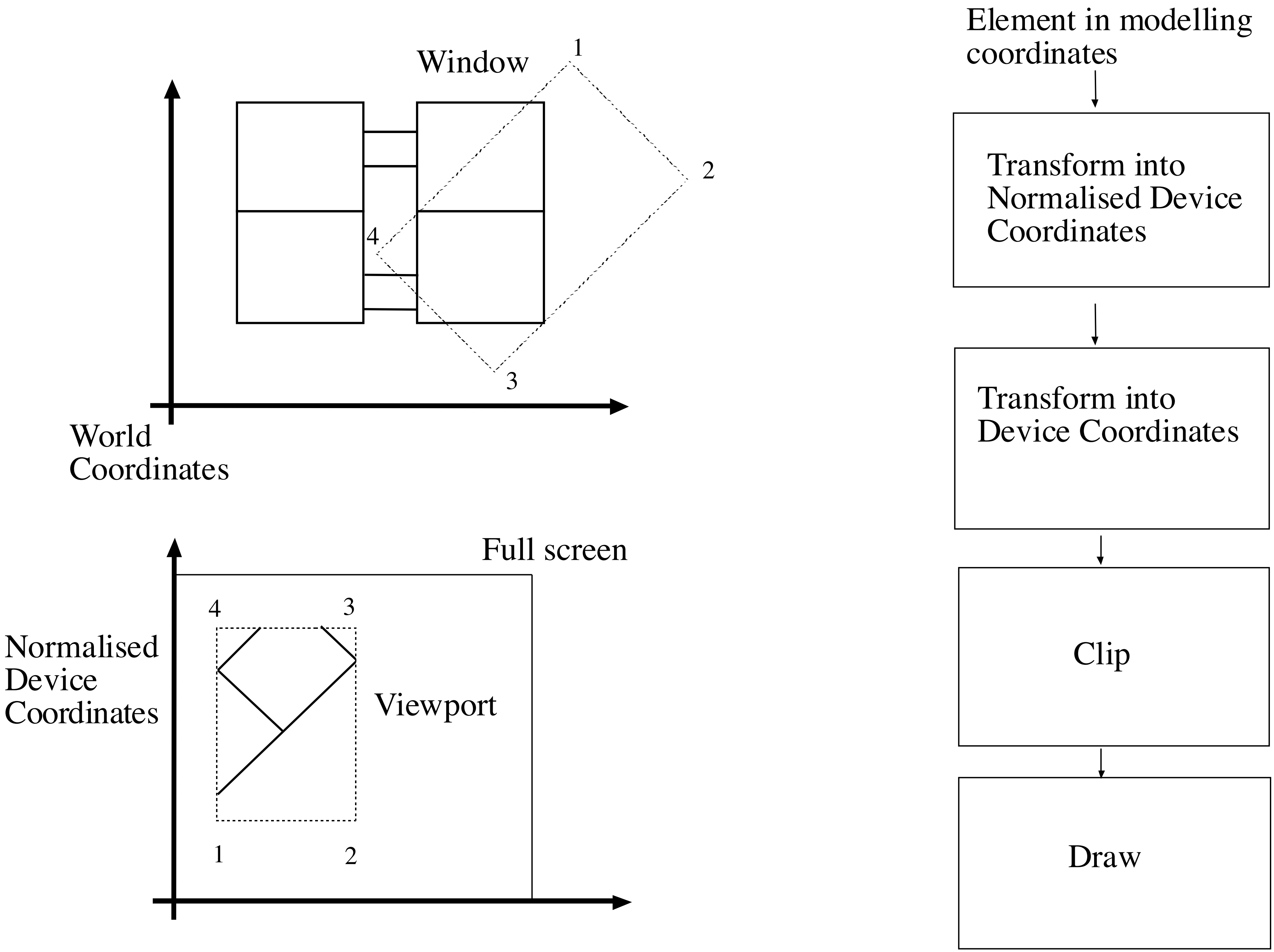
- Matrix for rotation about a point, P
- Problem--we only know how to rotate about the origin.
- Solution--translate to origin, rotate, and translate back

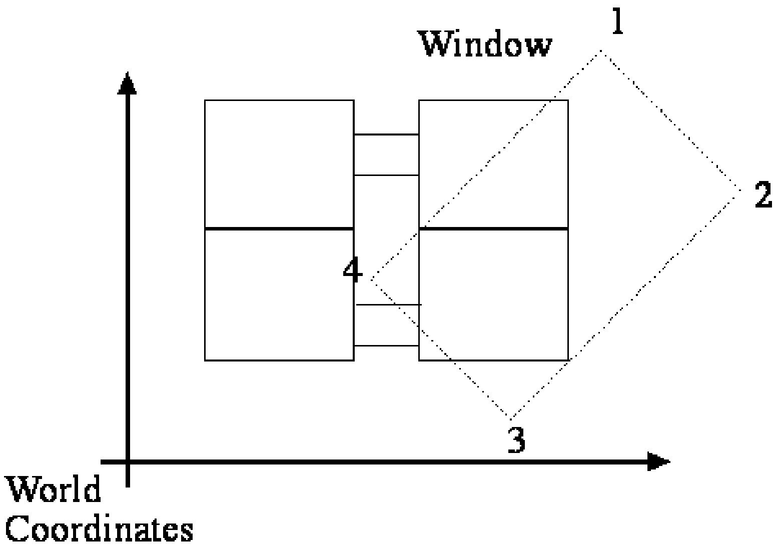
2D transformations (continued)

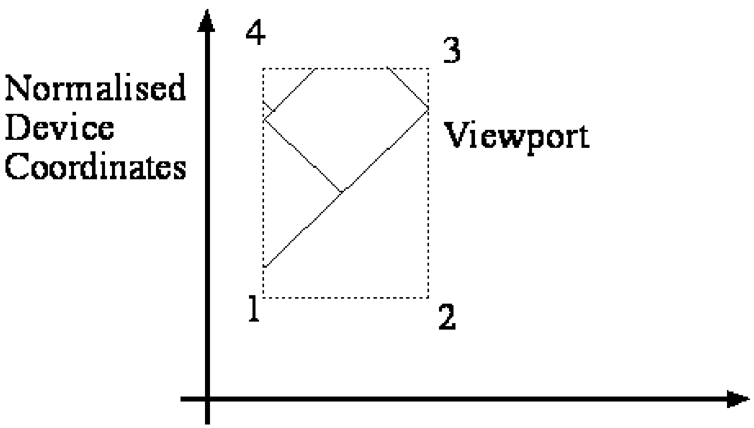
- The transformations discussed so far are invertable (why?). What are the inverses?

2D viewing

- Three coordinate systems are common in graphics
  - World coordinates or modeling coordinates - where the model is defined (meters, miles, etc.)
  - Normalized device coordinates; usually (0-1) in each variable.
  - Device coordinates: the actual coordinates of the pixels on the frame-buffer or the printer
- Need to construct transformations between coordinate systems
- Terminology:
  - window = region on drawing that will be displayed (rectangle)
  - viewport = region in NDC's/DC's where this rectangle is displayed (often simply entire screen).

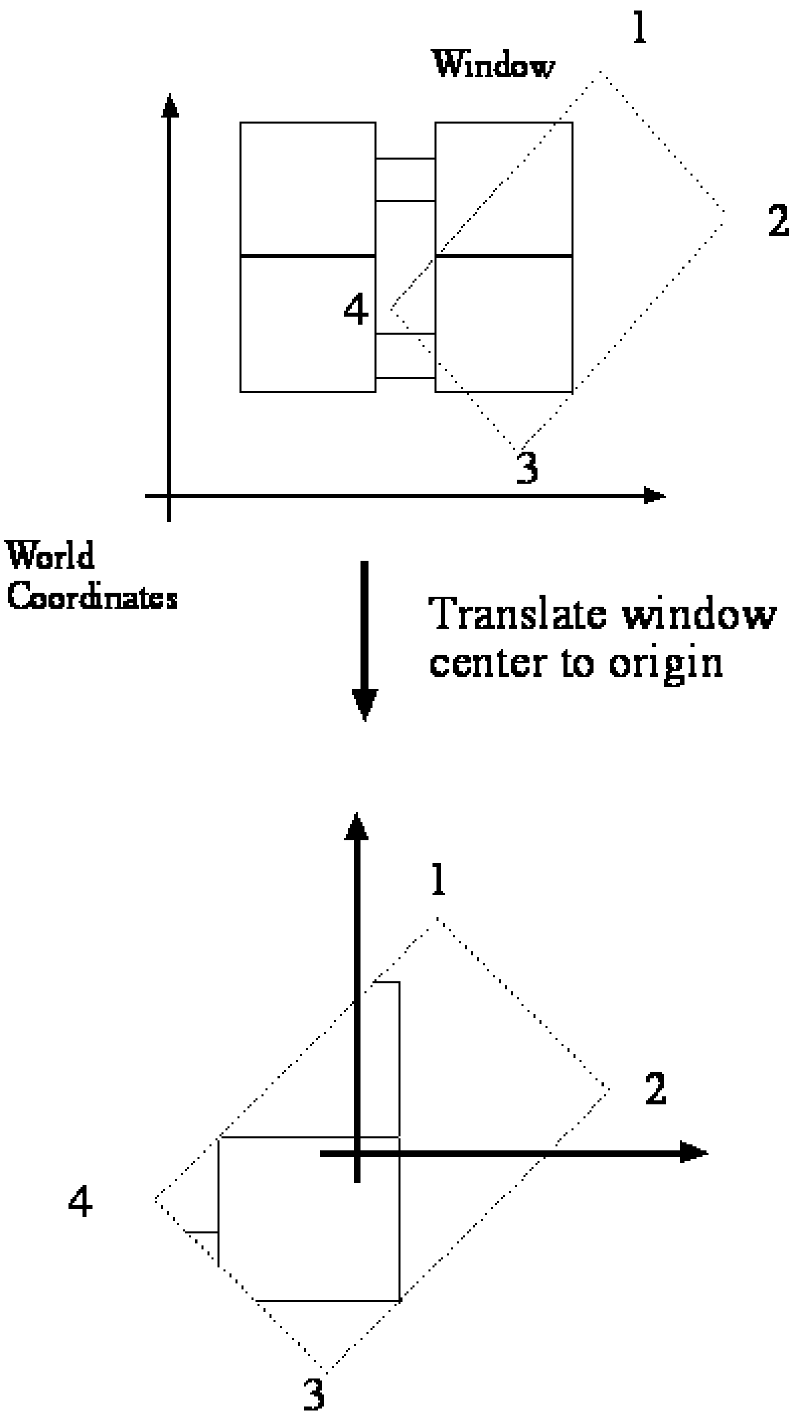

Window
1
2
3
4
World
Coordinates
Full screen
4
3
Normalised
Device
Coordinates
Viewport
1
2
Element in modelling
coordinates
Transform into
Normalised Device
Coordinates
Transform into
Device Coordinates
Clip
Draw





Determining the transform

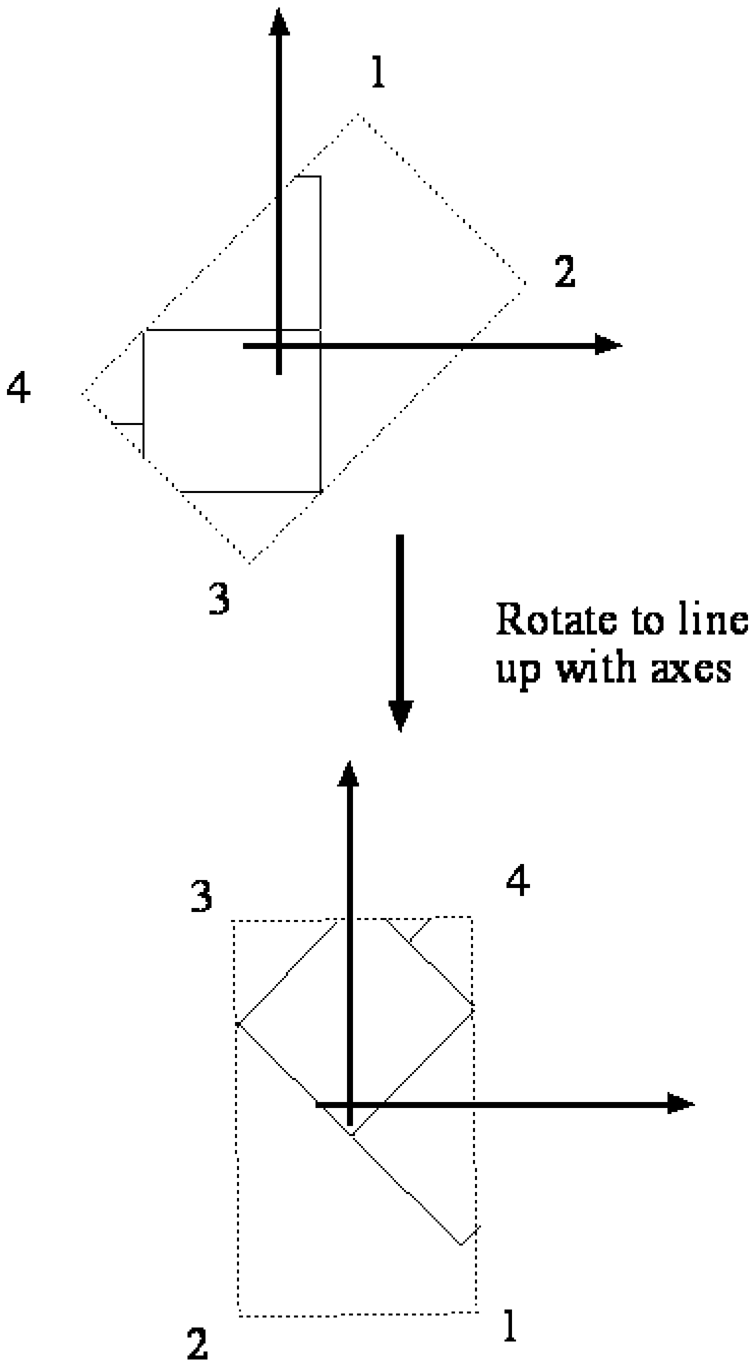
- **Plan A**: Consider this as a sequence of transformations in homogenous coords, then determine each element in closed form.
- **Plan B**: Compute numerically from point correspondences.



- write (wx_i, wy_i) for coordinates of i'th point on window
- translation is:

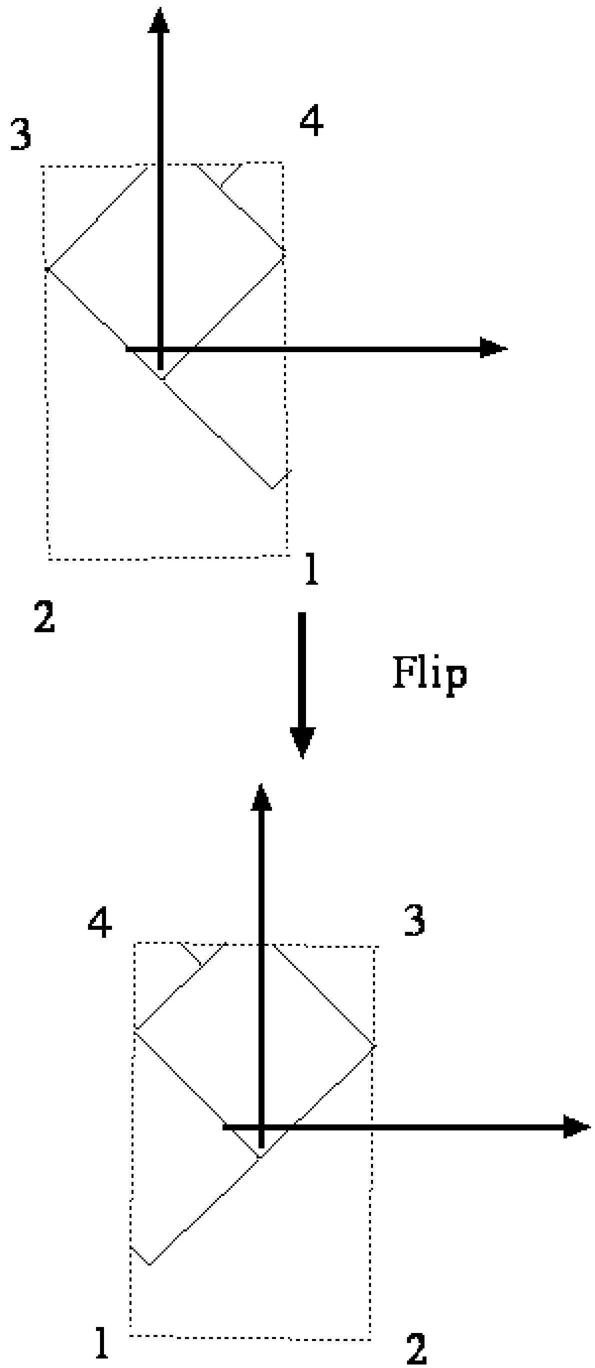
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \overline{wx} \\ 0 & 1 & \overline{wy} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(overbar denotes average over vertices, i.e., 1,2,3,4)



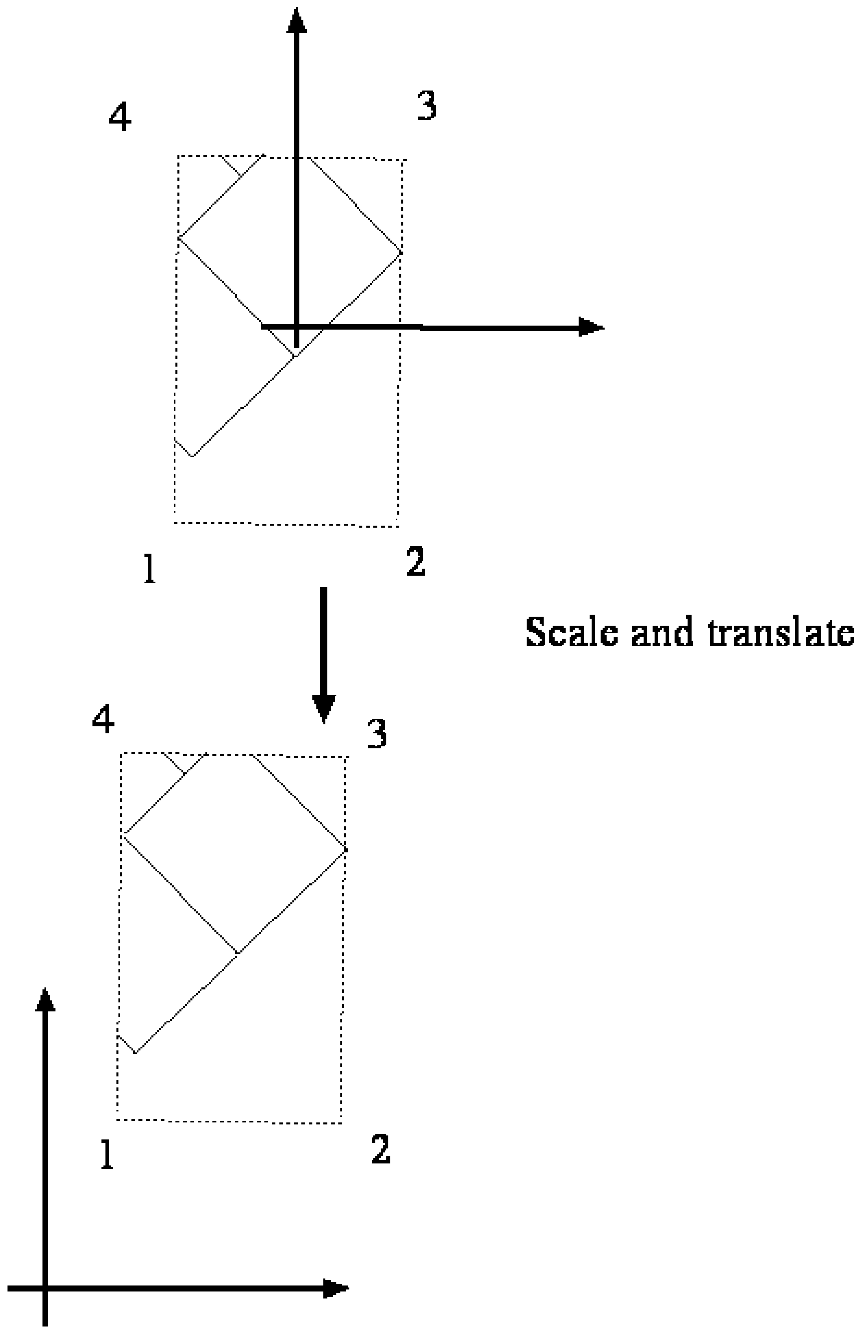
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(Need to compute theta)



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(Vertex order does not correspond, need to flip)



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{w_{new}}{w_{old}} & 0 & \overline{x}_{new} \\ 0 & \frac{h_{new}}{h_{old}} & \overline{y}_{new} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Notice that choice of new width, height, and center give translation to either normalized device coords, or to device coordinates

- Get overall transformation by multiplying transforms.
- This gives a single transformation matrix, whose elements are functions of window/viewport coordinates.

$$x' = M_{\text{(translate origin to viewport cog, scale)}}\ M_{\text{(flip)}}\ M_{\text{(rotate)}} M_{\text{(translate window cog->origin)}} x$$

NDC's/DC's

World coords

(cog==window center of gravity)

Plan B: Solve for the affine transformations

Details Optional

- Another approach to determining the whole transform for the pipeline; this is an affine transform.
- Matrix form:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

- Now assume that we know that $Mp_1=q_1$, $Mp_2=q_2$, $Mp_3=q_3$
- Quick way to determine transform, because this is the same as six linear equations, in six variables, which are the entries in the matrix (more details on next slide):

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ x_3 & y_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

Details

Details Optional

- $Mp_1=q_1$ gives first two rows
- $p_1 = (x_1, y_1, 1)^T$, $q_1 = (u_1, v_1, 1)^T$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} ax_1 + by_1 + c &= u_1 \\ dx_1 + ey_1 + f &= v_1 \end{aligned}}$$

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ x_3 & y_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$Mp_2=q_2$, $Mp_3=q_3$ give other rows