Homogenous Coordinates

- Represent 2D points by 3D vectors
- (x,y)-->(x,y,1)
- Now a multitude of 3D points (x,y,W) represent the same 2D point, (x/W, y/W, 1)
- Represent 2D transforms with 3 by 3 matrices
- Can now do translations
- Homogenous coordinates have other uses/advantages (later)

2D Translation in H.C.

$\mathbf{P}_{new} = \mathbf{P} + \mathbf{T}$

$(x', y') = (x, y) + (t_x, t_y)$

$$M = \begin{vmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{vmatrix}$$

2D Scale in H.C.

$$M = \begin{vmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

2D Rotation in H.C.

$$M = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Composition of Transformations

- If we use one matrix, M_1 for one transform and another matrix, M_2 for a second transform, then the matrix for the first transform followed by the second transform is simply $M_2 M_1$
- This generalizes to any number of transforms
- Computing the combined matrix **first** and then applying it to many objects, can save **lots** of computation

Composition Example

- Matrix for rotation about a point, P
- Problem--we only know how to rotate about the origin.

Composition Example

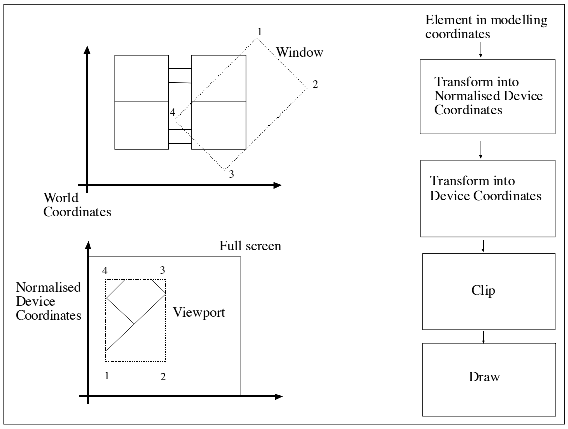
- Matrix for rotation about a point, P
- Problem--we only know how to rotate about the origin.
- Solution--translate to origin, rotate, and translate back

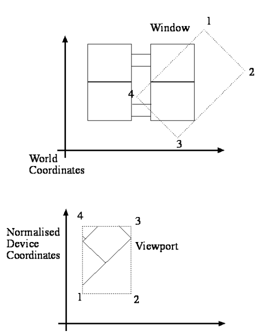
2D transformations (continued)

- The transformations discussed so far are invertable (why?). What are the inverses?

2D viewing

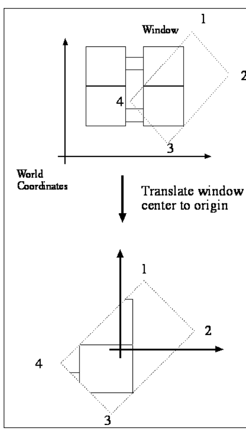
- Three coordinate systems are common in graphics
  - World coordinates or modeling coordinates - where the model is defined (meters, miles, etc.)
  - Normalized device coordinates; usually (0-1) in each variable.
  - Device coordinates: the actual coordinates of the pixels on the frame-buffer or the printer
- Need to construct transformations between coordinate systems
- Terminology:
  - window = region on drawing that will be displayed (rectangle)
  - viewport = region in NDC's/DC's where this rectangle is displayed (often simply entire screen).





Determining the transform

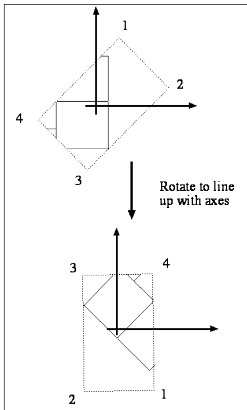
- **Plan A**: Consider this as a sequence of transformations in homogenous coords, then determine each element in closed form.
- **Plan B**: Compute numerically from point correspondences.



- write (wx_i, wy_i) for coordinates of i'th point on window
- translation is:

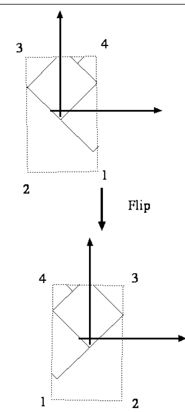
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\overline{wx} \\ 0 & 1 & -\overline{wy} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(overbar denotes average over vertices, i.e., 1,2,3,4)



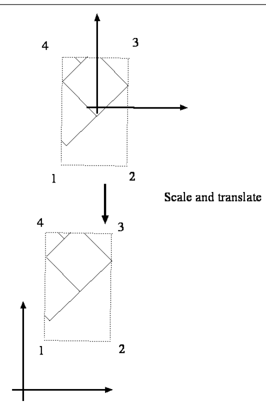
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(Need to compute theta)



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(Vertex order does not correspond, need to flip)



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{w_{new}}{w_{old}} & 0 & x_{new} \\ 0 & \frac{h_{new}}{h_{old}} & y_{new} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Notice that choice of new width, height, and center give translation to either normalized device coords, or to device coordinates

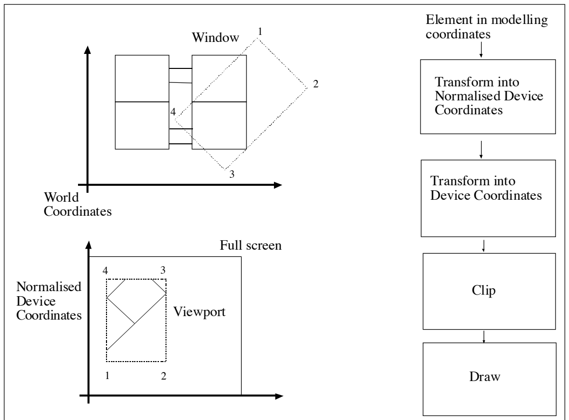
- Get overall transformation by multiplying transforms.
- This gives a single transformation matrix, whose elements are functions of window/viewport coordinates.

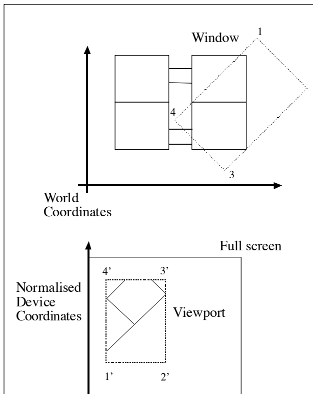
$$x' = M_{(\text{translate origin to viewport cog, scale})}\, M_{(\text{flip})}\, M_{(\text{rotate})} M_{(\text{translate window cog->origin})}\, x$$

NDC's/DC's

World coords

(cog==window center of gravity)





Plan B

Work **directly** from the fact that the transform we seek must map 1 to 1', 2 to 2', 3 to 3'.

Details Optional

Plan B: Solve for the affine transformation directly.

- We know that this is an “affine” transform.
- In particular, the matrix we seek is:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

More Details

Details Optional

- Consider the first mapping, $Mp_1=q_1$
- $p_1 = (x_1, y_1, 1)^T$, $q_1 = (u_1, v_1, 1)^T$

$$\begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} ax_1 + by_1 + c &= u_1 \\ dx_1 + ey_1 + f &= v_1 \end{aligned}}$$

More Details

Details Optional

Write

$$ax_1 + by_1 + c = u_1$$

As

$$x_1 a + y_1 b + 1 \bullet c + 0 \bullet d + 0 \bullet e + 0 \bullet f = u_1$$

Notice that this gives one equation in the six unknowns

More Details

Details Optional

Similarly, write

$$dx_1 + ey_1 + f = v_1$$

As

$$0 \bullet a + 0 \bullet b + 0 \bullet c + x_1 \bullet d + y_1 \bullet e + 1 \bullet f = u_1$$

Notice that this gives a second equation in the six unknowns

More Details

Details Optional

- $Mp_1=q_1$ gives first two rows
- $p_1 = (x_1, y_1, 1)^T$, $q_1 = (u_1, v_1, 1)^T$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} ax_1 + by_1 + c &= u_1 \\ dx_1 + ey_1 + f &= v_1 \end{aligned}}$$

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ x_3 & y_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$Mp_2=q_2$, $Mp_3=q_3$ give other rows

More Details

Details Optional

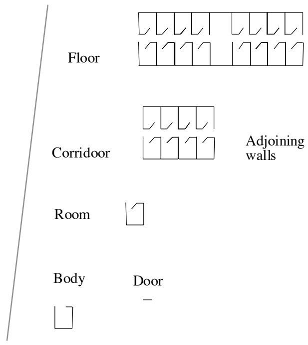
Final representation of six equations in six unknowns

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ x_3 & y_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

This can be solved using standard methods

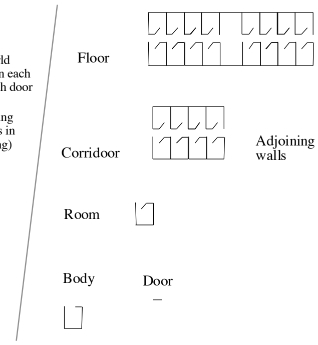
Hierarchical modeling

- Consider constructing a complex 2d drawing: e.g. an animation showing the plan view of a building, where the doors swing open and shut.

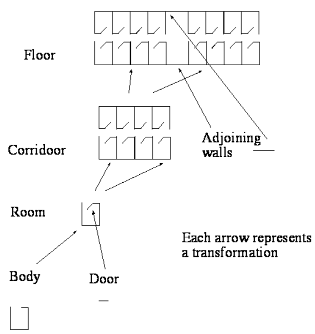


Hierarchical modeling

- Options:
  - specify everything in world coordinate frame; but then each room is different, and each door moves differently.
  - Exploit similarities by using repeated copies of models in different places (instancing)



Hierarchical modeling



Hierarchical modeling

- Model form
  - Directed acyclic graph.
  - Each node consists of 0 or more objects (lines, polygons, etc).
  - Each edge is a transformation
- There can be many edges joining two nodes (e.g. in the case of the corridor - many copies of the same room model, each transformed differently).
- Every graphics API supports hierarchies - some directly (meaning you have to learn a language to express the model) some indirectly with a matrix stack

Hierarchical modeling

Write the transformation from door coordinates to room coordinates as:

$$T_{room}^{door}$$

Then to render a door, use the transformation:

$$T_{device}^{world} T_{floor}^{corridoor} T_{corridoor}^{room} T_{room}^{door}$$

To render a body, use the transformation:

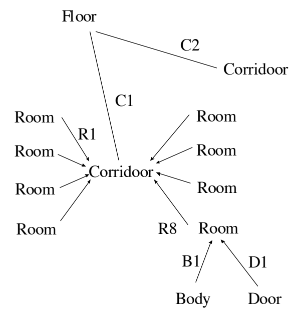
$$T_{device}^{world} T_{floor}^{corridoor} T_{corridoor}^{room} T_{room}^{body}$$

Matrix stacks and rendering

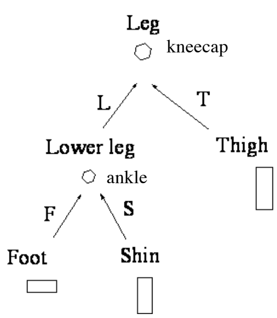
- Matrix stack:
  - Stack of matrices used for rendering
  - Applied in sequence.
  - Pop=remove last matrix
  - Push=append a new matrix
  - In previous example, body-device transformation comes from door-device transformation by popping door-room and pushing body-room

Matrix stacks and rendering

- Algorithm for rendering a hierarchical model:
  - stack is T_{device}^{root}
  - render (root)
- Recursive definition of render (node)
  - if node has object, render it
  - for each child:
    - push transformation
    - render (child)
    - pop transformation



- Now to render door on first room in first corridor, stack looks like: W C1 R1 D1
- For efficiency we would store “running” products, IE, the stack contains: W, W*C1, W*C1*R1, W*C1*R1*D1.
- We do not need two copies of corridor, or 16 copies of body; we render one copy using 16 different transformations. This is known as instancing
- Animation requires care: if D1 is a single function of time, all doors will swing open and closed at the same time.



- Stack is W
- render kneecap
- Stack is W L
- render ankle
- Stack is W L F
- render foot
- Stack is W L S
- render shin
- Stack is W T
- render thigh