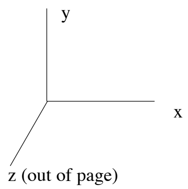
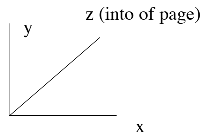
Transformations in 3D

- Right hand coordinate system (conventional, i.e., in math)



- In graphics a LHS is sometimes also convenient (Easy to switch between them--later).



Transformations in 3D

- Homogeneous coordinates now have four components - traditionally, (x, y, z, w)
  - ordinary to homogeneous: (x, y, z) -> (x, y, z, 1)
  - homogeneous to ordinary: (x, y, z, w) -> (x/w, y/w, z/w)
- Again, translation can be expressed as a multiplication.

Transformations in 3D

- Translation:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D transformations

- Anisotropic scaling:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Shear (one example):

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotations in 3D

- 3 degrees of freedom
- Orthogonal, det(R)=1
- We can easily determine formulas for rotations about each of the axes
- For general rotations, there are many possible representations—we will use a **sequence** of rotations about coordinate axes.
- Sign of rotation follows the Right Hand Rule--point thumb along axis in direction of increasing ordinate--then fingers curl in the direction of positive rotation).

Rotations in 3D

- About x-axis

$$M = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Rotations in 3D

- About y-axis

$$M = \begin{vmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Rotations in 3D

- About z-axis

$$M = \begin{vmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Commuting transformations

- If A and B are matrices, does AB=BA? Always? Ever?
- What if A and B are restricted to particular transformations?
- What about the 2D transformations that we have studied?
- How about if A and B are restricted to be on of the three specific 3D rotations just introduced, such as rotation about the Z axis?

Demo

Commuting transformations

- If A and B are matrices, does AB=BA? Always? Ever?
- What if A and B are restricted to particular transformations?
- What about the 2D transformations that we have studied?
- How about if A and B are restricted to be on of the three specific 3D rotations just introduced, such as rotation about the Z axis?

Answer: In general AB != BA (matrix multiplication is not commutative). But if A and B are either translations or scalings, then multiplication is commutative. The same applies to rotations restricted to be about one of the 3 axis in 3D.

Rotations in 3D

- About X axis

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

- 90 degrees about X axis?

Rotations in 3D

- About X axis

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- 90 degrees about X axis

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations in 3D

- About Y axis

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- 90 degrees about Y-axis?

Rotations in 3D

- About Y axis

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- 90 degrees about Y axis

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations in 3D

- 90 degrees about X then Y

$$\underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Y rot}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{X rot}} = ?$$

Rotations in 3D

- 90 degrees about X then Y

$$\underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Y rot}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{X rot}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations in 3D

- 90 degrees about X then Y

$$\underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Y rot}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{X rot}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- 90 degrees about Y then X

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{X rot}} \underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Y rot}} = ?$$

Rotations in 3D

- 90 degrees about X then Y

$$\begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Y rot   X rot

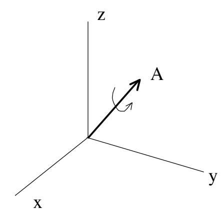
≠

- 90 degrees about Y then X

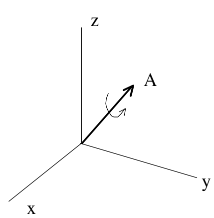
$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

X rot   Y rot

Rotation about an arbitrary axis

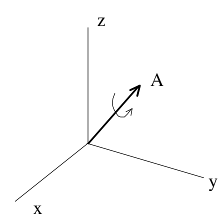


Rotation about an arbitrary axis



Strategy--rotate A to Z axis, rotate about Z axis, rotate Z back to A.

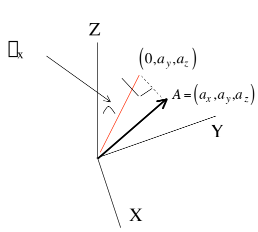
Rotation about an arbitrary axis



Tricky part:
rotate A to Z axis

Two steps.
1) Rotate about x to xz plane
2) Rotate about y to Z axis.

Rotation about an arbitrary axis

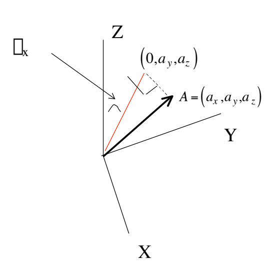


Tricky part:
rotate A to Z axis

Two steps.
1) Rotate about X to xz plane
2) Rotate about Y to Z axis.

As A rotates into the xz plane, its projection (shadow) onto the YZ plane (red line) rotates through the same angle which is easily calculated.

Rotation about an arbitrary axis



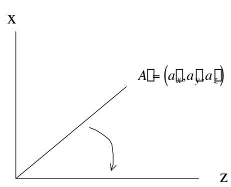
$$d = \sqrt{a_y^2 + a_z^2}$$

$$\sin\theta_x = a_y / d$$

$$\cos\theta_x = a_z / d$$

No need to compute angles, just put sines and cosines into rotation matrices

Rotation about an arbitrary axis



Apply $R_x(\theta_x)$ to A and renormalize to get A'

$R_y(\theta_y)$ should be easy, but note that it is clockwise.

Rotation about an arbitrary axis

Final form is

$$R_x(-\theta_x)R_y(-\theta_y)R_z(\theta_z)R_y(\theta_y)R_x(\theta_x)$$