Transformations in 3D

- Right hand coordinate system (conventional, i.e., in math)
- In graphics a LHS is sometimes also convenient (Easy to switch between them--later).
Transformations in 3D

• Homogeneous coordinates now have four components - traditionally, (x, y, z, w)
  – ordinary to homogeneous: (x, y, z) -> (x, y, z, 1)
  – homogeneous to ordinary: (x, y, z, w) -> (x/w, y/w, z/w)

• Again, translation can be expressed as a multiplication.
Transformations in 3D

- Translation:

\[
\begin{bmatrix}
1 & 0 & 0 & tx \\
0 & 1 & 0 & ty \\
0 & 0 & 1 & tz \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
**3D transformations**

- **Anisotropic scaling:**

\[
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix} =
\begin{bmatrix}
sx & 0 & 0 & 0 \\
0 & sy & 0 & 0 \\
0 & 0 & sz & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

- **Shear (one example):**

\[
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & a & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]
Rotations in 3D

- 3 degrees of freedom
- Orthogonal, det(R)=1
- We can easily determine formulas for rotations about each of the axes
- For general rotations, there are many possible representations—we will use a sequence of rotations about coordinate axes.
- Sign of rotation follows the Right Hand Rule--point thumb along axis in direction of increasing ordinate--then fingers curl in the direction of positive rotation).
Rotations in 3D

- About x-axis

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 & 0 \\
0 & -\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
Rotations in 3D

- About y-axis

\[ M = \begin{bmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \]
Rotations in 3D

- About z-axis

\[
M = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 
\end{bmatrix}
\]
Commuting transformations

• If A and B are matrices, does AB=BA? Always? Ever?
• What if A and B are restricted to particular transformations?
• What about the 2D transformations that we have studied?
• How about if A and B are restricted to be on of the three specific 3D rotations just introduced, such as rotation about the Z axis?
Demo
Commuting transformations

- If $A$ and $B$ are matrices, does $AB = BA$? Always? Ever?
- What if $A$ and $B$ are restricted to particular transformations?
- What about the 2D transformations that we have studied?
- How about if $A$ and $B$ are restricted to be on of the three specific 3D rotations just introduced, such as rotation about the Z axis?

**Answer:** In general $AB \neq BA$ (matrix multiplication is not commutative). But if $A$ and $B$ are either translations or scalings, then multiplication is commutative. The same applies to rotations restricted to be about one of the 3 axis in 3D.
Rotations in 3D

• About X axis

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos q & \sin q & 0 \\
0 & -\sin q & \cos q & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

• 90 degrees about X axis?
Rotations in 3D

- About X axis

\[
\begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & \cos q & \sin q & 0 \\
0 & \sin q & \cos q & 0 \\
0 & 0 & 0 & 1
\end{vmatrix}
\]

- 90 degrees about X axis

\[
\begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{vmatrix}
\]
Rotations in 3D

• About Y axis

\[
\begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta \\
\end{pmatrix}
\]

• 90 degrees about Y-axis?
Rotations in 3D

- About Y axis
  
  $\begin{bmatrix}
  \cos(q) & 0 & \sin(q) & 0 \\
  0 & 1 & 0 & 0 \\
  \sin(q) & 0 & \cos(q) & 0 \\
  0 & 0 & 0 & 1
  \end{bmatrix}$

- 90 degrees about Y axis
  
  $\begin{bmatrix}
  0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1
  \end{bmatrix}$
Rotations in 3D

- 90 degrees about X then Y

\[
\begin{pmatrix}
0 & 0 & \overline{1} & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & \overline{1} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
= ?

Y rot  \quad X rot
Rotations in 3D

- 90 degrees about X then Y

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{array}
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}
= \begin{array}{ccc}
0 & \mathbf{1} & 0 \\
0 & 0 & \mathbf{1} \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}
\]

Y rot \quad X rot
Rotations in 3D

- 90 degrees about X then Y

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Y rot \quad X rot

- 90 degrees about Y then X

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= ?
\]

X rot \quad Y rot
Rotations in 3D

- 90 degrees about X then Y

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Y rot                  X rot

- 90 degrees about Y then X

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

X rot                  Y rot

≠
Rotation about an arbitrary axis
Rotation about an arbitrary axis

Strategy--rotate A to Z axis, rotate about Z axis, rotate Z back to A.
Rotation about an arbitrary axis

Tricky part:
rotate A to Z axis

Two steps.
1) Rotate about x to xz plane
2) Rotate about y to Z axis.
Rotation about an arbitrary axis

As \( A \) rotates into the xz plane, its projection (shadow) onto the YZ plane (red line) rotates through the same angle which is easily calculated.

Tricky part:
- rotate \( A \) to Z axis

Two steps.
1) Rotate about X to xz plane
2) Rotate about Y to Z axis.
Rotation about an arbitrary axis

\[ d = \sqrt{a_y^2 + a_z^2} \]

\[ \sin \theta_x = \frac{a_y}{d} \]

\[ \cos \theta_x = \frac{a_z}{d} \]

No need to compute angles, just put sines and cosines into rotation matrices
Rotation about an arbitrary axis

Apply $R_x(\Box_x)$ to $A$ and renormalize to get $A'$.

$R_y(\Box_y)$ should be easy, but note that it is clockwise.
Rotation about an arbitrary axis

Final form is

$$R_x(\square x)R_y(\square y)R_z(\square z)R_y(\square y)R_x(\square x)$$