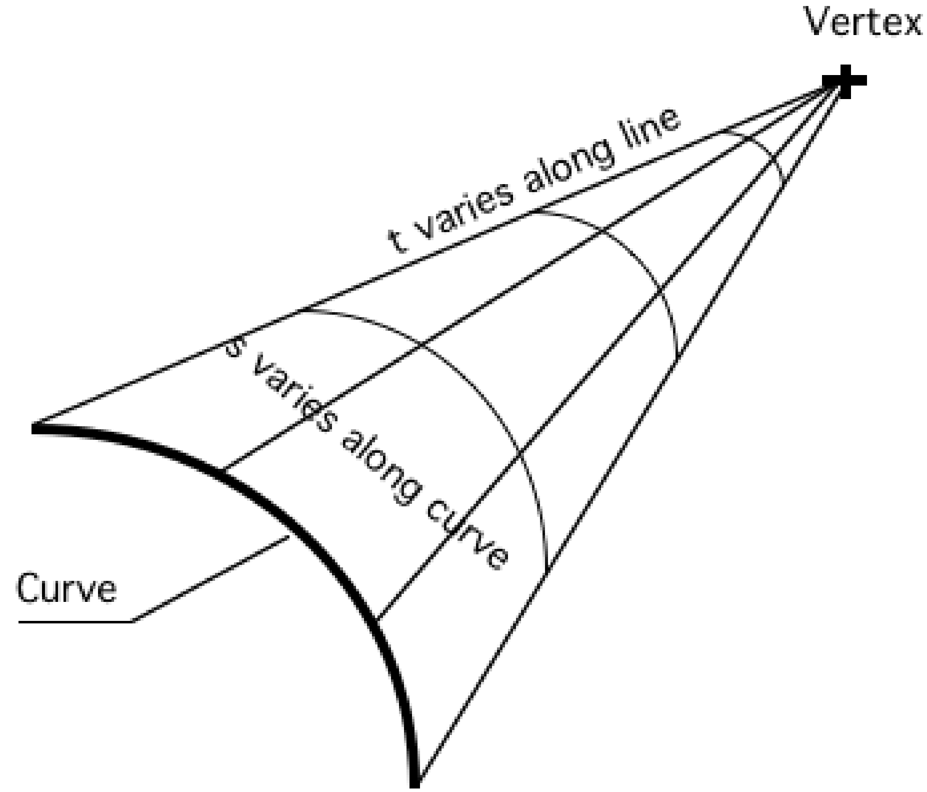
Cones

- From every point on a curve, construct a line segment through a single fixed point in space - the vertex
- Curve can be space or plane curve, but shouldn't pass through the vertex

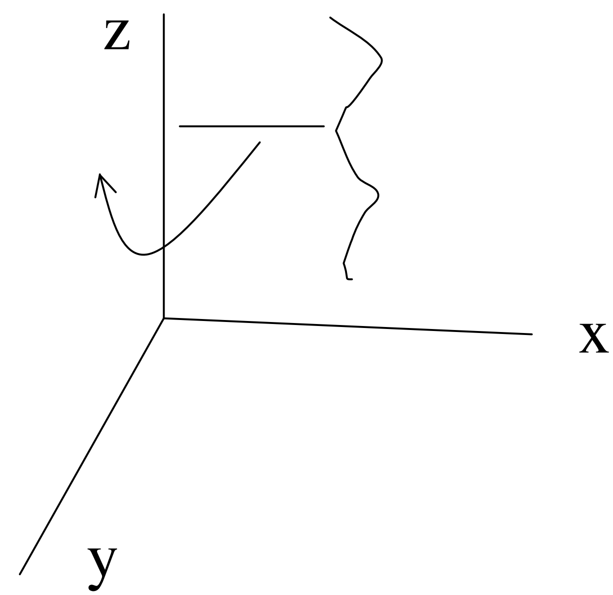


$$(x(s,t), y(s,t), z(s,t)) = (1-t)(x_c(s), y_c(s), z_c(s)) + t(v_0, v_1, v_2)$$

Surfaces of revolution

- Plane curve + axis
- "spin" plane curve around axis to get surface
- Choice of plane is arbitrary, choice of axis affects surface
- In the example to the right, curve is on x-z plane, axis is z axis.
- So curve is $(x_c(s), z_c(s))$

Parametric formula?

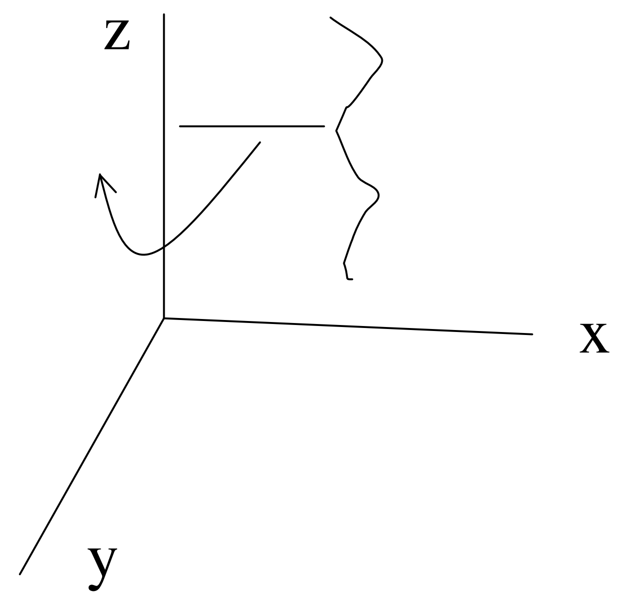


Surfaces of revolution

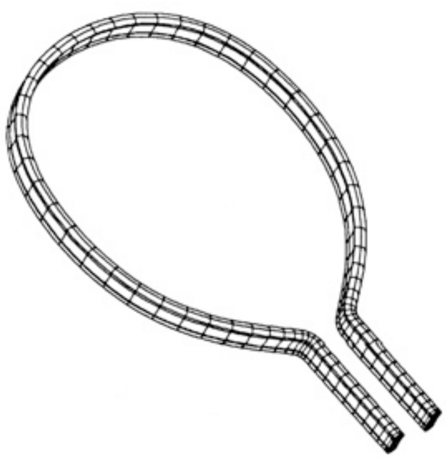
- Plane curve + axis
- "spin" plane curve around axis to get surface
- Choice of plane is arbitrary, choice of axis affects surface
- In the example to the right, curve is on x-z plane, axis is z axis. (Think of $x_c(s)$ as a radius)

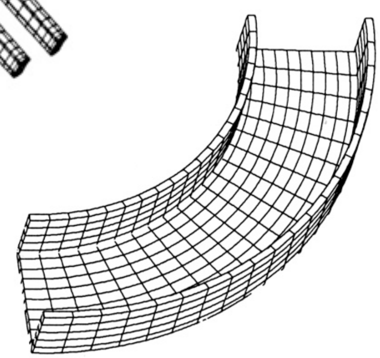
$$(x(s,t), y(s,t), z(s,t)) =$$

$$(x_c(s)\cos(t), x_c(s)\sin(t), z_c(s))$$



Sweeps/Generalized Cylinders





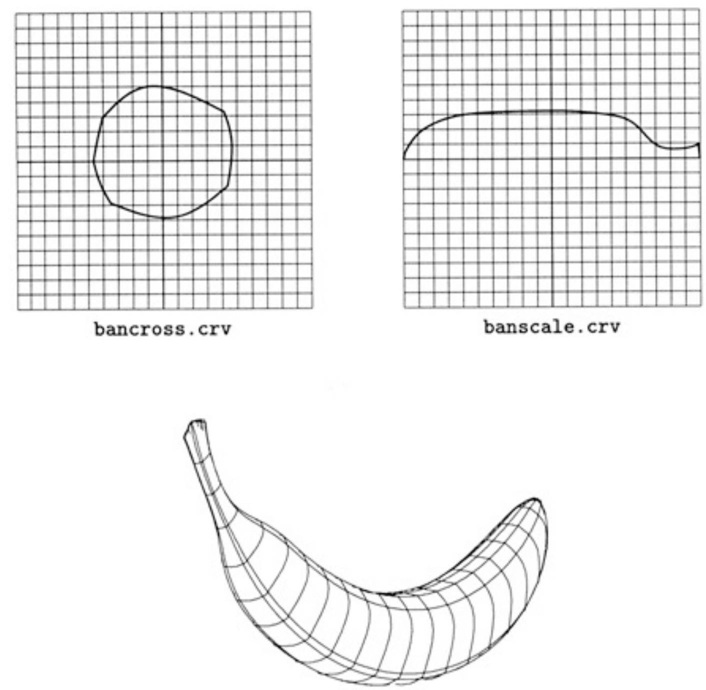
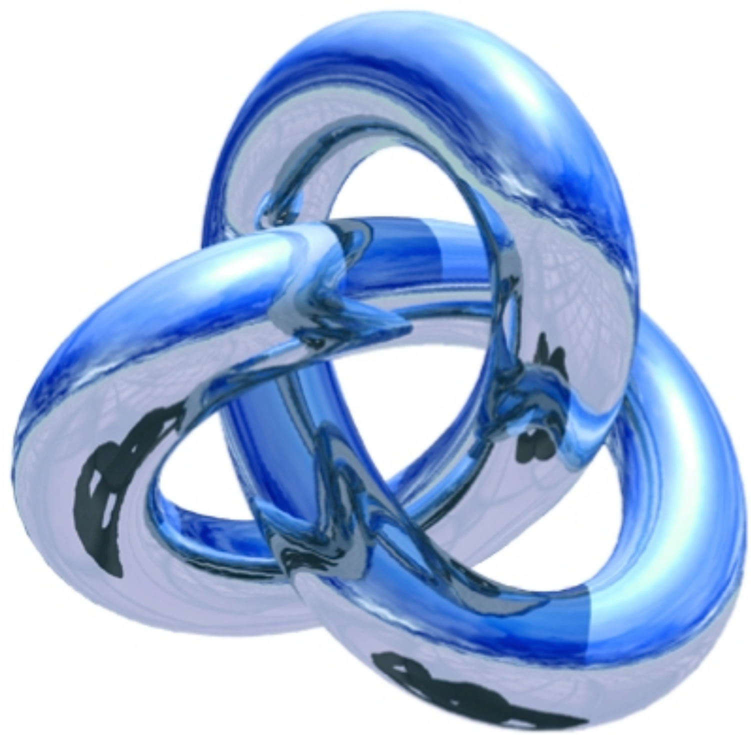


Figure 3.8: Banana example. A banana is represented by an affine transformation surface. The cross section is scaled, translated along z from -1 to 1, and rotated around the y axis.
□

[Synder 92, via CMU course page]

Sweeps/Generalized Cylinders



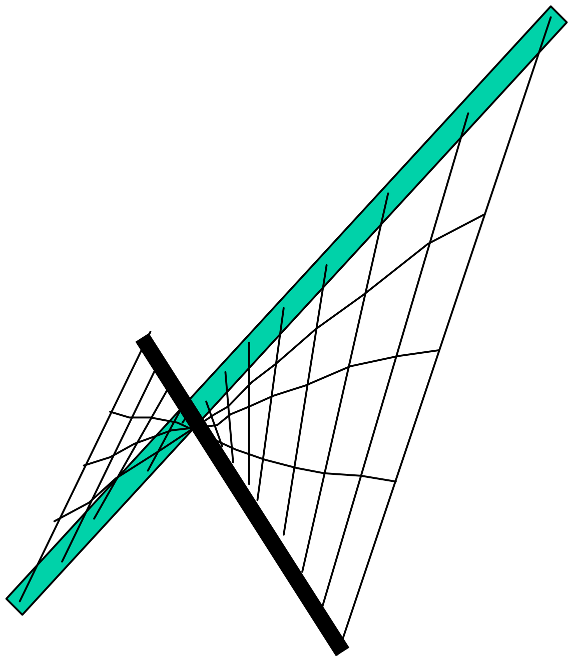
MetaCreations, via CMU course page

Ruled surfaces -1

- Popular, because it's easy to build a curved surface out of straight segments - e.g. pavilions, etc.
- Take two space curves, and join corresponding points—same s parameter value—with line segment.
- Even if space curves are lines, the surface is usually curved.

Ruled Surfaces - 2

Easy to explain,
hard to draw!



Ruled surfaces -3

Parameterized form

$$(x(s,t), y(s,t), z(s,t)) =$$
$$(1-t)(x_1(s), y_1(s), z_1(s)) +$$
$$t(x_2(s), y_2(s), z_2(s))$$

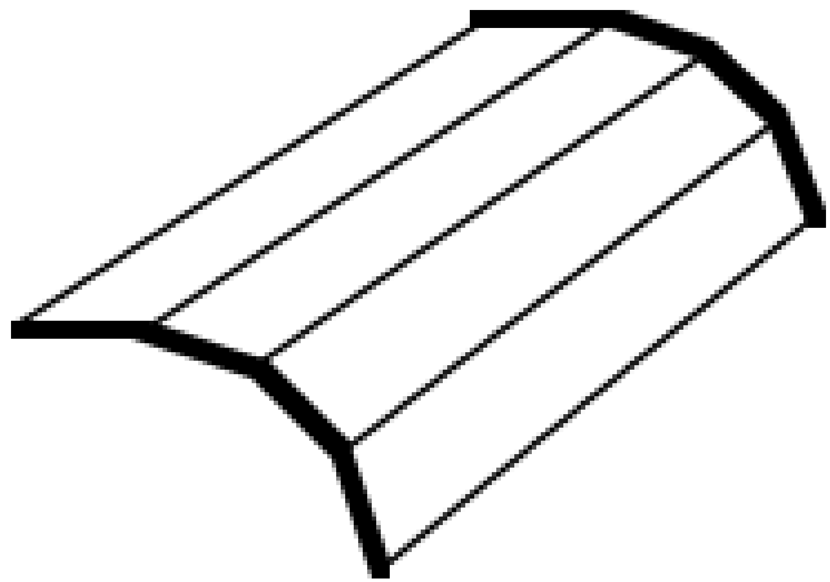
Normals

- Normal is cross product of tangent in t direction and s direction.

$$\left(\frac{\delta x}{\delta t},\frac{\delta y}{\delta t},\frac{\delta z}{\delta t}\right)\times\left(\frac{\delta x}{\delta s},\frac{\delta y}{\delta s},\frac{\delta z}{\delta s}\right)$$

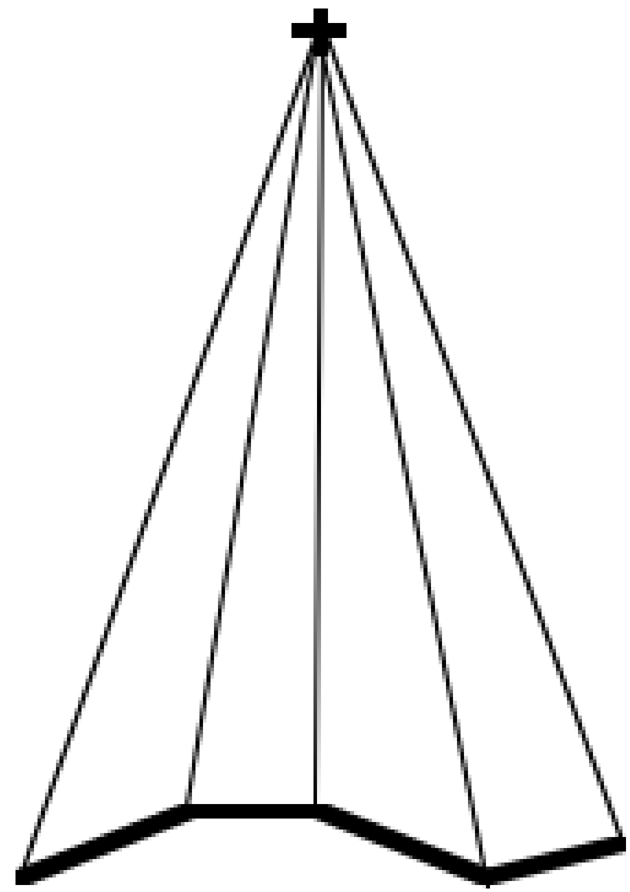
Rendering

- Cylinders: small steps along curve, straight segments along t generate polygons; exact normal is known.



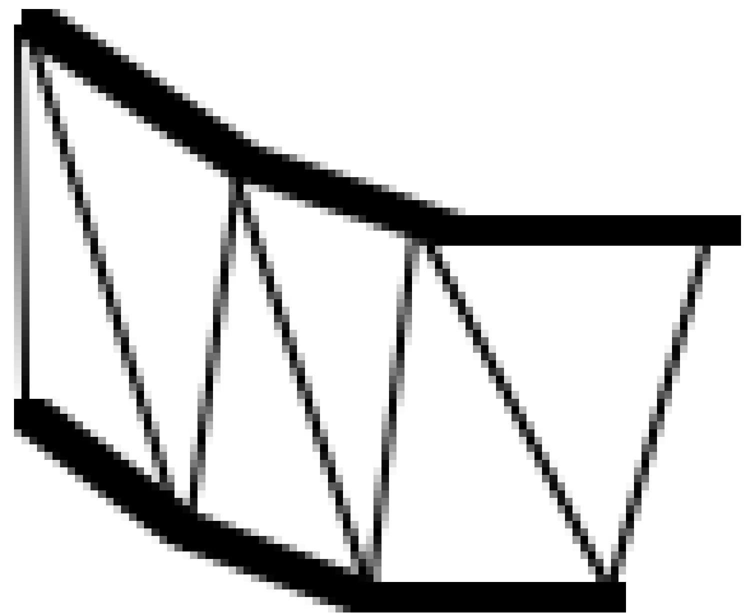
Rendering

- Cone: small steps in s generate straight edges, join with vertex to get triangles, normals known exactly except at vertex.



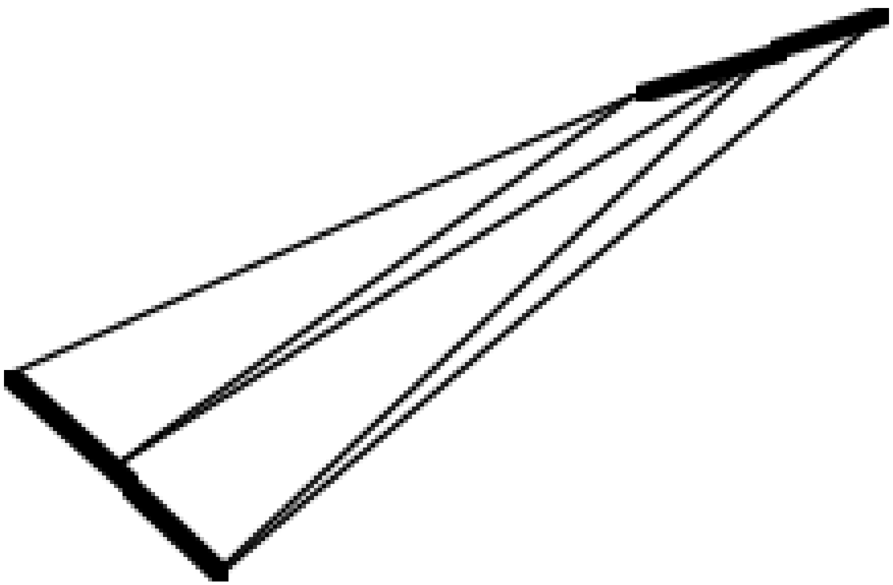
Rendering

- Surface of revolution: small steps in s generate strips, small steps in t along the strip generate edges; join up to form triangles. Normals known exactly.



Rendering

- Ruled surface: steps in s generate polygons, join opposite sides to make triangles - otherwise "non planar polygons" result. Normals known exactly.
- **Must** understand why rectangular sections do not work!



Specifying Curves from Points

- Want to modulate curves via “control” points.
- Strategy depends on application. Possibilities:
  - Force a polynomial of degree N-1 through N points (Lagrange interpolate)
  - Specify a combination of “anchor” points and derivatives (Hermite interpolate)
  - Other “blends” (Bezier, B-splines)--more useful than Lagrange/Hermite

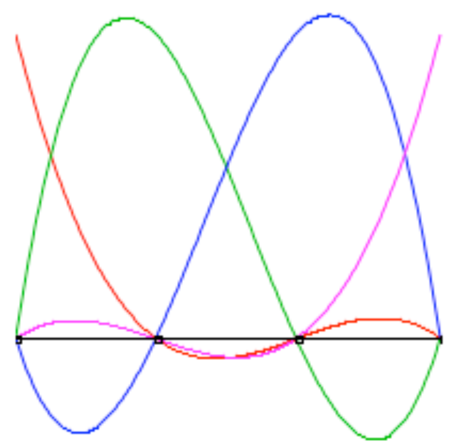
Specifying Curves from Points-II

- Issues:
  - Local versus global control
  - Higher polynomial degree versus stitching lower order polynomials together (stitching-->local control)
  - Continuity of curve and derivatives (geometric, parametric)
  - Polynomials verses other forms
  - Polynomial degree (usually 3--fewer is not flexible enough, and higher gives hard to control wiggles).
  - It is relatively easy to fit a curve through points in explicit form, but we will use parametric form as it more useful in graphics.

Lagrange Interpolate (degree 3)

- Want a parametric curve that passes through (interpolates) four points.
- Use the points to combine four Lagrange polynomials (blending functions)
- As the parameter goes through each of 4 particular values, one blending function is 1, and the other 3 are zero.

$$\sum_{i \in \text{points}} p_i \phi_i^{(l)}(t)$$

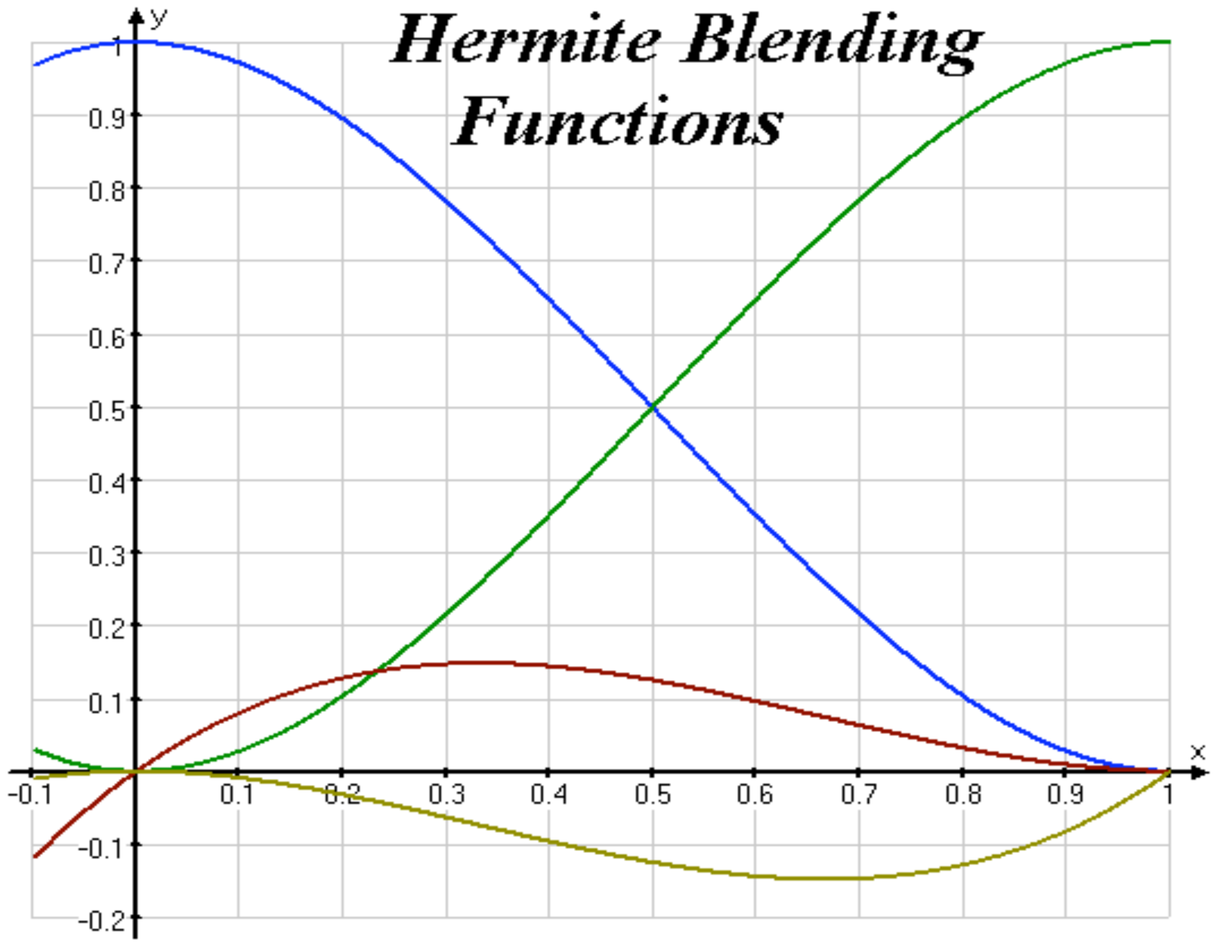


Hermite (H&B, page 426)

- Hermite interpolate
  - Curve passes through specified points **and** has specified derivatives at those points.

- Standard degree 3 case: 2 points, 2 derrivatives at those points
- 4 functions of degree 3, two each of two kinds
  - one at an endpoint, zero at the other, AND derrivative is zero at both
  - derivative is one at an endpoint and zero at others, AND value is zero at the endpoints.

$$\sum_{i \in \text{points}} p_i \phi_i^{(h)}(t) + \sum_{i \in \text{points}} v_i \phi_i^{(hd)}(t)$$



From www.cs.virginia.edu/~gfx/Courses/ 2002/Intro.fall.02

Blended curves

- Assume degree 3
- Includes Hermite, Bézier and others

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = C * T(t) = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Blended curves

- General pattern: Decompose the matrix C into two factors
  - One factor encodes the “control” or data points
  - The second factor is the blending functions

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = [G_1 \quad G_2 \quad G_3 \quad G_4] \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Hermite

- Geometry matrix
  - First two columns are endpoints
  - Next two columns are derivatives at those points
- Blending function matrix

$$M_H = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Where does this come from?

Skipped in 06

Hermite

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = [G_1 \quad G_2 \quad G_3 \quad G_4] M_H \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Can solve for M_H using x(t) only (or y(t), or z(t))

$$x(t) = [G_{1x} \quad G_{2x} \quad G_{3x} \quad G_{4x}] * M_H * T(t)$$

Hermite

Skipped in 06

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = [G_1 \quad G_2 \quad G_3 \quad G_4] M_H \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Can solve for M_H using x(t) only (or y(t), or z(t))

$$x(t) = [G_{1x} \quad G_{2x} \quad G_{3x} \quad G_{4x}] * \underbrace{M_H * T(t)}$$

Vector of blending functions; compare with previous representation.

$$\sum_{i \in \text{points}} p_i \phi_i^{(h)}(t) + \sum_{i \in \text{points}} v_i \phi_i^{(hd)}(t)$$

Skipped in 06

Hermite

$$x(t) = [G_{1x} \quad G_{2x} \quad G_{3x} \quad G_{4x}]\ M_H * \mathbf{T}(t)$$

We have

$x(0) = G_{1x}$ so $M_H * T(0) = ?$
$x(1) = G_{2x}$ so $M_H * T(1) = ?$
$x'(0) = G_{3x}$ so $M_H * T'(0) = ?$
$x'(1) = G_{4x}$ so $M_H * T'(1) = ?$

Skipped in 06

Hermite

$$x(t) = [G_{1x} \quad G_{2x} \quad G_{3x} \quad G_{4x}]\ M_H * \mathbf{T}(t)$$

We have

$x(0) = G_{1x}$ so $M_H * T(0) = [1\ 0\ 0\ 0\ ]^T$
$x(1) = G_{2x}$ so $M_H * T(1) = [0\ 1\ 0\ 0\ ]^T$
$x'(0) = G_{3x}$ so $M_H * T'(0) = [0\ 0\ 1\ 0\ ]^T$
$x'(1) = G_{4x}$ so $M_H * T'(1) = [0\ 0\ 0\ 1\ ]^T$

Hermite

Skipped in 06

x(0) = G_{1x} so M_H * **T**(0) = [1 0 0 0 $]^T$
x(1) = G_{2x} so M_H * **T**(1) = [0 1 0 0 $]^T$
x'(0) = G_{3x} so M_H * **T**'(0) = [0 0 1 0 $]^T$
x'(1) = G_{4x} so M_H * **T**'(1) = [0 0 0 1 $]^T$

So

$$M_H * [\mathbf{T(0)} \quad \mathbf{T(1)} \quad \mathbf{T'(0)} \quad \mathbf{T'(1)}] = I$$

So

$$M_H = [\mathbf{T(0)} \quad \mathbf{T(1)} \quad \mathbf{T'(0)} \quad \mathbf{T'(1)}]^{-1}$$

Hermite

Skipped in 06

$$\mathbf{T}(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\mathbf{T}'(t) = ?$$

Hermite

Skipped in 06

$$\mathbf{T}(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} \qquad \mathbf{T}'(t) = \begin{bmatrix} 3t^2 \\ 2t^2 \\ 1 \\ 0 \end{bmatrix}$$

So, **T**(0), **T**(1), **T'**(0), and **T'**(1) are?

Hermite

Skipped in 06

$$\mathbf{T}(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} \qquad \mathbf{T}'(t) = \begin{bmatrix} 3t^2 \\ 2t^2 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{T(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{T}(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{T}'(\mathbf{0}) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{T}'(1) = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Hermite

Skipped in 06

Recall that

$$M_H * \begin{bmatrix}\mathbf{T(0)} & \mathbf{T(1)} & \mathbf{T'(0)} & \mathbf{T'(1)}\end{bmatrix} = I$$

And thus we seek

$$M_H = \begin{bmatrix}\mathbf{T(0)} & \mathbf{T(1)} & \mathbf{T'(0)} & \mathbf{T'(1)}\end{bmatrix}^{-1}$$

$$M_H = \begin{vmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix}^{-1}$$

Hermite

Skipped in 06

Finally

$$M_H = \begin{vmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix}^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$