Convolution and Correlation

- Correlation
  - New signal by moving desired mask/template around.
  - New values follow the mask
- Convolution
  - Filter is the response to a unit bar (or delta function)
  - Convolution gives response to the entire signal
- Can switch from one to the other by flipping the filter
  - $h_{corr}(x) = h_{conv}(-x)$
  - $h_{corr}(x, y) = h_{conv}(-x, -y)$
- No difference if filter is symmetric

Image Filtering Preliminaries

- Denote the image by F (to follow the book).
- Represent weights as a second image, H (the kernel).
- Pretend that images are padded to infinity with zeros (so sums don't need limits).
- To shift a function f (x,y) up and to the right by (a,b)
  - f (x-a, y-b)

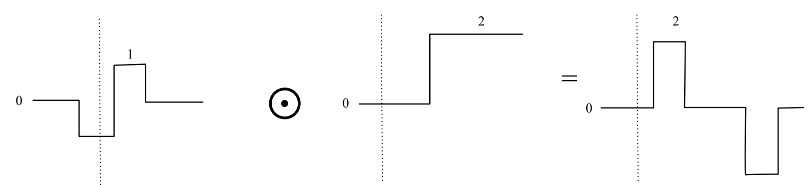
Correlation

- Denote by $\odot$
- Then the definition of discrete 2D correlation is:

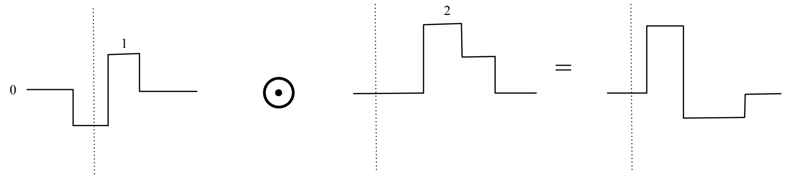
$$R_{i,j} = \sum_{u,v} H_{u-i,v-j} F_{u,v}$$

Puts filter on (i,j)

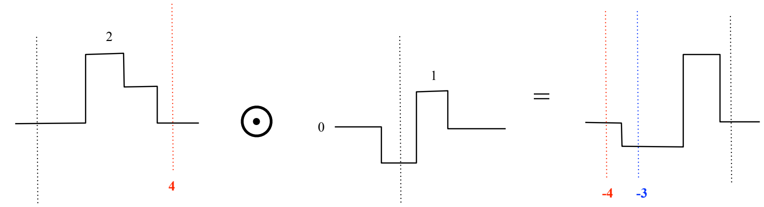
Correlation example one



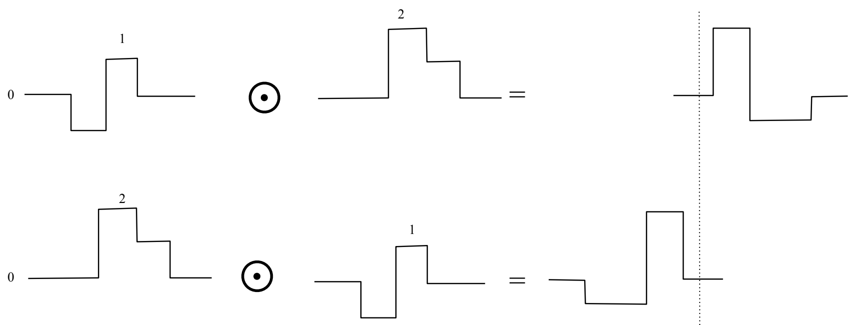
Correlation example two



Correlation example three (switch H and F)



Correlation (H ⊙ F versus F ⊙ H)



Correlation is not always commutative
(example one is, example two is not)

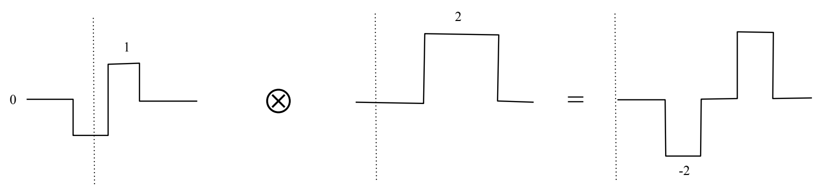
Convolution

- Denote by $\otimes$
  - Others symbols include * (for 1D) and ** (for 2D).
- Matlab "conv" (1D) and "conv2" (2D)
- The definition of discrete 2D convolution is:

$$R_{i,j} = \sum_{u,v} H_{i-u,j-v} \; F_{u,v}$$

- Notice weird order of indices (includes the flips)

Convolution example



Properties of $R_{i,j} = \sum_{u,v} H_{i-u,j-v} \; F_{u,v}$

- Commutative (unlike correlation)

  Compare $H_{i-u}F_u$ and $F_{i-u'}H_{u'}$ when subscripts of F are the same.

  In other words, $u = i - u'$. Plug this into the first H, to get $i - u = u'$.

  Note that the pairings are always the same. (QED)

- Associative (Can save CPU time!)

  $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

- Linear $(aA + bB) \otimes C = a(A \otimes C) + b(B \otimes C)$

  and

  $C \otimes (aA + bB) = a(C \otimes A) + b(C \otimes B)$

Properties of $R_{i,j} = \sum_{u,v} H_{i-u,j-v} \; F_{u,v}$

- Commutative (contrast with correlation)
- **Associative** (Can save CPU time!)
- Linear
- Output is a s**hift-invariant** function of the input (i.e. shift the input image two pixels to the left, the output is shifted two pixels to the left)
- Converse of above is true: If a system is linear and shift invariant, then it is a convolution.

Shift invariant linear systems (§7.2)

- Shift invariant
  - Shift in the input means we simply shift the output
  - Example: Optical system response to a point of light
    - Light moves from center to edge, so does its image
- Linear shift invariant
  - Can compute the output due to complex input, based on the response to a single point input
    - Discrete version---function *box*(x,y) is zero everywhere except at (x',y') where is is 1.
    - Continuous version---delta function
- f(x,y) is a linear combination of shifted versions of *box*(x',y')

Rewrite f(i,j) as a sum over its natural basis

$$f(i,j) = \sum_u \sum_v box(i-u, j-v)\, f(u,v)$$

Box shifted by (u,v). Note subtraction!

Example, if $f(i,j) = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

$$f(i,j) = 2 \bullet \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 3 \bullet \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} + 4 \bullet \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + 5 \bullet \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$$

Also, if $box(u,v) = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$

$$f(i,j) = 2 \bullet box(i-0, j-0) + 3 \bullet box(i-0, j-1) + 4 \bullet box(i-1, j-0) + 5 \bullet box(i-1, j-1)$$

$$f(i,j) = \sum_{u=0}^{1} \sum_{v=0}^{1} box(i-u, j-v)\, f(u,v)$$

(reverse multiplcation order to follow convention.

Before we derive convolution ...

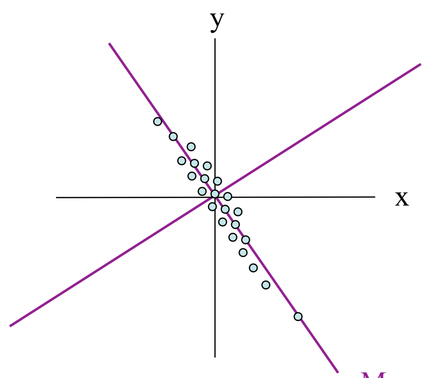
- Note that we are thinking of images as a linear combination of simple units (a "basis").
- You **have** seen this before

$$\mathbf{v} = (3,-2,5) = 3 \bullet \hat{\mathbf{i}} + (-2) \bullet \hat{\mathbf{j}} + 5 \bullet \hat{\mathbf{k}}$$

$$e^x = 1 \bullet x^0 + 1 \bullet x^1 + \left(\frac{1}{2}\right) \bullet x^2 + \left(\frac{1}{2 \bullet 3}\right) \bullet x^3 + \ \ldots$$

- The notion of basis is an important abstraction because rewriting images (etc) with respect to different bases can provide insight and/or solves problems.

Another example



More useful basis for this data than the usual (1,0) and (0,1).

Rewrite f(i,j) as a sum over its natural basis

$$f(i,j) = \sum_u \sum_v box(i-u, j-v)\, f(u,v)$$

Box shifted by (u,v). Note subtraction!

Given that

$$\text{Response}(box(i,j)) = h(i,j)$$

Shift invariance means that

$$\text{Response}(box(i-u, j-v)) = h(i-u, j-v)$$

Linearity means we can bring the response inside the sum.

$$\text{Response}(f(i,j)) = R_{ij} = \sum \sum \ h(i-u, j-v)\, f(u,v)$$

(Convolution by h)

In more detail

$$response\{f(i,j)\} = response\left\{\sum_u \sum_v box(i-u, j-v) \bullet f(u,v)\right\}$$

$$= \sum_u \sum_v response\{box(i-u, j-v) \bullet f(u,v)\} \qquad \text{(linearity)}$$

$$= \sum_u \sum_v response\{box(i-u, j-v)\} \bullet f(u,v) \qquad \text{(linearity)}$$

$$= \sum_u \sum_v h(i-u, j-v) \bullet f(u,v) \qquad \text{(shift-invariant)}$$

In the last step, we have used the fact that we can get the response to the shifted filter by shifting the response.

This derives convolution in terms of responses to a unit impulse function (here denoted by box()).

Response as sum of basis functions (§7.2)

- The response is linear combination of shifted versions of the kernel
- The weights are the values of the function being convolved
- The shifted versions of the kernels form a basis over which the result image is constructed
- Thinking of an image as a weighted sum over a basis is a generally useful idea—e.g., Fourier transforms.

Response as sum of basis functions (§7.2)

- Linear shift invariant systems explains the "flip" is in the previous formula
  - Shifting rewrote the function values so that the kernel was flipped
  - Convolution by h() implies a basis of shifted, flipped, h()
  - Getting the right answer **requires** flips if the kernel is not symmetric

Correlation

- Similar to convolution (no flips)
- Implements convolution (if a flip is used) or vice versa
- Finds things in images that "look like" the kernel
- The kernel is also referred to as a "mask", especially in application oriented discussion (both in convolution and correlation).

2D convolution example (from MathWorks website)

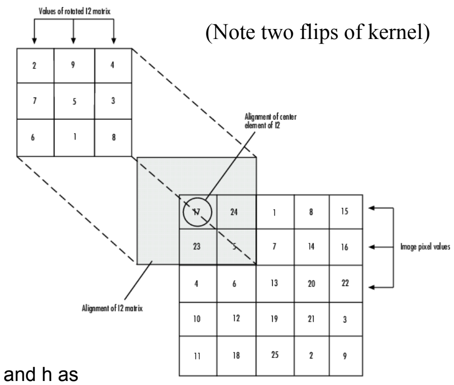
For example, suppose the image is

```
A = [17  24   1   8  15
     23   5   7  14  16
      4   6  13  20  22
     10  12  19  21   3
     11  18  25   2   9 ]
```

and the convolution kernel is

```
h = [8  1  6
     3  5  7
     4  9  2]
```

R(1,1) = 5*17+3*24+1*23+8*5



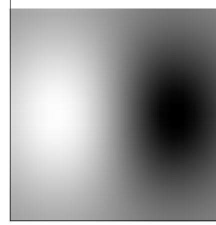
(Note two flips of kernel)

To do the complete convolution, set A and h as above in Matlab, and do conv2(A,h,'same'). Try also conv2(A,h) --- make sure you understand the difference!

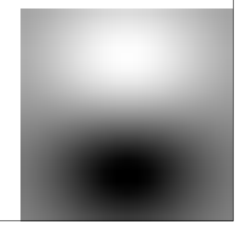
Filters are templates

- Applying a filter at some **point** can be seen as taking a dot-product between the image and some vector
- Filtering the image yields a set of dot products

- Useful intuition
  - Filters look like the effects they are intended to find.
  - Filters find effects that look like them.
  - Remember to flip your filter if you are implementing correlation using convolution.



Filters for steps in X (left) and Y (right). The step in X goes from high-to-low. Convolving with it finds high-to-low steps due to the flip.



Normalized correlation

- Think of filters of a dot product
  - **problem**: brighter parts give bigger results even if the structure is same (often not what you want)
  - **normalized** correlation output is filter output, divided by root sum of squares of values over which filter lies

$$\frac{\mathbf{h} \bullet \mathbf{f}}{|\mathbf{f}|}$$

(**f** is limited to where h is non zero)

  - Can think in terms of angle between vectors. Recall

$$\cos(\theta) = \frac{\mathbf{h} \bullet \mathbf{f}}{|\mathbf{h}||\mathbf{f}|}$$

(|**h**| is not relevant to this problem)

Normalized correlation

- Some tricks of the trade
  - Consider template filters that have zero response to a constant region (helps reduce response to irrelevant background).