Finding M (goal one) (§3)

Find M from an image of calibration object. The equation relating world coordinates to image coordinates is:

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = M \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = MP$$

If we identify enough non-degenerate points whose *world coordinates are known* then we can estimate M from their *location in the image*.

Specifically we have points in space, P, and corresponding observed image coordinates, u=U/W and v=V/W

(§2.2.2, §3.2.1)

We have $\begin{pmatrix} U \\ V \\ W \end{pmatrix} = M\mathbf{P}$

Write $M = \begin{vmatrix} \mathbf{m_1} \\ \mathbf{m_2} \\ \mathbf{m_3} \end{vmatrix}$ Where $\mathbf{m}_i$ are row vectors

Thus
$$U = \mathbf{m_1} \cdot \mathbf{P}$$
$$V = \mathbf{m_2} \cdot \mathbf{P}$$
$$W = \mathbf{m_3} \cdot \mathbf{P}$$

(§2.2.2, §3.2.1)

From the previous slide

$$U = \mathbf{m_1} \cdot \mathbf{P}$$
$$V = \mathbf{m_2} \cdot \mathbf{P}$$
$$W = \mathbf{m_3} \cdot \mathbf{P}$$

So **each** point, **i**, gives two equations (§2.2.2, §3.2.1)

$$u_i = \frac{\mathbf{m}_1 \cdot \mathbf{P}_i}{\mathbf{m_3} \cdot \mathbf{P}_i} \qquad v_i = \frac{\mathbf{m}_2 \cdot \mathbf{P}_i}{\mathbf{m_3} \cdot \mathbf{P}_i}$$

Which become

$$(\mathbf{m_1} - u_i \mathbf{m_3}) \cdot \mathbf{P}_i = 0$$
$$(\mathbf{m}_2 - v_i \mathbf{m_3}) \cdot \mathbf{P}_i = 0$$

(§2.2.2, §3.2.1)

We have linear equations for the components of M

The components of the matrix M are the *variables* in linear equations

Represent M by a vector $\mathbf{m} = \begin{pmatrix} \mathbf{m}_1^{\mathbf{T}} \\ \mathbf{m}_2^{\mathbf{T}} \\ \mathbf{m}_3^{\mathbf{T}} \end{pmatrix}$

Note that our camera matrix, M, is the unknown so we want to make it a vector in some matrix equation (where something else is going to be the matrix)---standard thing to do.

(§2.2.2, §3.2.1)

We are representing the matrix M by a vector $\mathbf{m} = \begin{pmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \\ \mathbf{m}_3^T \end{pmatrix}$

Now rewrite $(\mathbf{m}_1 - u_i \mathbf{m}_3) \cdot \mathbf{P}_i = 0$ as $\begin{pmatrix} \mathbf{P}_i^T & 0 & -u_i \mathbf{P}_i^T \end{pmatrix} \mathbf{m} = 0$

$(\mathbf{m}_2 - v_i \mathbf{m}_3) \cdot \mathbf{P}_i = 0$ as $\begin{pmatrix} 0 & \mathbf{P}_i^T & -v_i \mathbf{P}_i^T \end{pmatrix} \mathbf{m} = 0$

Thus every point leads to two rows of a matrix P.

(§2.2.2, §3.2.1)

From previous slide, each point gives two rows of a matrix P

$$\begin{pmatrix} \mathbf{P}_i^T & 0 & -u_i \mathbf{P}_i^T \end{pmatrix} \mathbf{m} = 0$$
$$\begin{pmatrix} 0 & \mathbf{P}_i^T & -v_i \mathbf{P}_i^T \end{pmatrix} \mathbf{m} = 0$$

So, in general, the 2n by 12 matrix P is:

$$\begin{pmatrix} \mathbf{P}_1^T & & -u_1 \mathbf{P}_1^T \\ & \mathbf{P}_1^T & -v_1 \mathbf{P}_1^T \\ \cdots & \cdots & \cdots \\ \mathbf{P}_i^T & & -u_i \mathbf{P}_i^T \\ & \mathbf{P}_i^T & -v_i \mathbf{P}_i^T \\ \cdots & \cdots & \cdots \\ \mathbf{P}_n^T & & -u_n \mathbf{P}_n^T \\ & \mathbf{P}_n^T & -v_n \mathbf{P}_n^T \end{pmatrix}$$

(§2.2.2, §3.2.1)

So, we want to solve P**m**=**0** for **m**, where P is 2n by 12

This problem is a bit tricky

Clearly **m**=**0** is a solution (degenerate solution)

There must be another solution (if we believe our imaging model)

If **m** is a solution, then a scalar multiple of **m** is also (homogeneity)

So, we solve P**m**=**0** under the constraint that |**m**|=1

If n>6, then this typically will not have a solution due to error (over-constrained)

To simultaneously deal with this problem, AND to use the information from multiple points, we find a "best" solution, using more than 6 points.

Math aside, #4

Homogenous linear least squares

Recall the second problem inspired by our camera calibration problem:

Solve $U\mathbf{x} = \mathbf{0}$ subject to $|\mathbf{x}| = 1$

We will sketch the solution briefly

(still §3.1.1)

Details optional

Homogenous linear least squares

Because we solve $U\mathbf{x} = \mathbf{0}$ as best we can, the error vector is $U\mathbf{x}$

The squared error is then

$$(U\mathbf{x})^{\mathbf{T}}(U\mathbf{x}) = \mathbf{x}^{\mathbf{T}}(U^{\mathbf{T}}U)\mathbf{x}$$

Since $U^{\mathbf{T}}U$ is symmetric it has an eigenvalue decomposition (diagonalization) with real eigenvalues

Recall that a matrix A has an eigenvector, $\mathbf{e}$, with eigenvalue λ if
$A\mathbf{e} = \lambda\mathbf{e}$

Diagonalization: $U^{\mathbf{T}}U = V\Lambda V^{T}$ where V is an orthonormal basis made of the eigenvectors, $\mathbf{e}_i$, of $U^{\mathbf{T}}U$, and Λ is a diagonal matrix of the eigenvalues

Details optional

Homogenous linear least squares

Critically, since U^TU it is positive semidefinite, the eigenvalues are **non−negative**

Recall that a matrix A is positive semidefinite if $\mathbf{x}^T A\mathbf{x}$ is never negative.
(Clearly U^TU it is positive semidefinite because $\mathbf{x}^T U^T U\mathbf{x}$ is $|U\mathbf{x}|^2$

Details optional

Further technical comments

If the model is good, then U will **approximate** a matrix of deficient column rank because there should exist a non-zero **x** that solves U**x**=**0**.

We force the solution process to embody the assumption that the fact that U^TU *appears* to be of full rank is due to measurement error. This assumption helps separate the part of U that is due to errors from the part that is due to the model.

This why we say that U^TU is **semi-**positive definite, and *not* positive definite. We assume that there is a solution to Ux=0, that is distinctly non-zero .

(A matrix, A, is positive definite if, $\mathbf{x}^T A\mathbf{x}$ is never negative, and $\mathbf{x}^T A\mathbf{x} = \mathbf{0}$ means that $\mathbf{x}=\mathbf{0}$.)

Details optional

Homogenous linear least squares

Since U^TU it is positive semidefinite, the eigenvalues are **non−negative**

(From two slides back)

We will write them as $\lambda_i = \sigma_i^2$.

Note: The book (at least my copy) uses λ_i^2 in the equation at the top of page 41 which is confusing. The coefficients, normally denoted, λ_i are in fact equal to the **square** of the "singular values of U", which usually are denoted by σ_i

Details optional

Homogenous linear least squares

We can write $\mathbf{x}$ in terms of the eigenvector basis:
$\mathbf{x} = \sum u_i \mathbf{e}_i$ where $\sum u_i^2 = 1$ (why?)

(Because $\mathbf{x}^T\mathbf{x} = \sum u_j \mathbf{e}_j^T \sum u_i \mathbf{e}_i = \sum \sum u_i u_j \mathbf{e}_j^T \mathbf{e}_i = \sum u_i^2$)

Details optional

Homogenous linear least squares

The error is $\mathbf{x}^T\left(V\Lambda V^T\right)\mathbf{x} = \left(\mathbf{x}^T V\right)\Lambda\left(V^T\mathbf{x}\right)$

What is $V^T\mathbf{x}$?

Recall that $\mathbf{x} = \sum u_i \mathbf{e}_i$

And that the columns of V are the eigenvectors $\mathbf{e}_i$

So the elements of $V^T\mathbf{x}$ are u_i

Details optional

Homogenous linear least squares

The error is $\mathbf{x}^T\left(V\Lambda V^T\right)\mathbf{x} = \left(\mathbf{x}^T V\right)\Lambda\left(V^T\mathbf{x}\right)$

The elements of $V^T\mathbf{x}$ are u_i

So the error is $\sum u_i^2 \lambda_i = \sum u_i^2 \sigma_i^2$

Details optional

Homogenous linear least squares

From the previous slide the error to be minized is $\sum u_i^2 \sigma_i^2$

We are stuck with the values σ_i^2 and $\sum u_i^2 = 1$

The best we can do is to set $u_i = 1$
for the minimum value of $\lambda_i = \sigma_i^2$ and zero for the others.

Homogenous linear least squares

The best we can do is to set $u_i = 1$
for the minimum value of $\lambda_i = \sigma_i^2$ and zero for the others.

$$\mathbf{x} = \sum u_i \mathbf{e}_i$$

Set $u_{i*} = 1$, all other $u_i = 0$

So, $x^* = e_{i*}$, where λ_{i^*} = is minimum

Important

Thus the minimum is reached when x is set to the eigenvector corresponding to the minimum eigenvalue of $U^T U$