Rotations in 3D

- 3 degrees of freedom
- Orthogonal, with $\det(R)=1$
- We can easily determine formulas for rotations about each of the axes
- For general rotations, there are many possible representations—we will use a sequence of rotations about coordinate axes.
- Sign of rotation follows the Right Hand Rule—point thumb along axis in direction of increasing ordinate—then fingers curl in the direction of positive rotation).

Rotations in 3D

- About x-axis

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Rotations in 3D

- About y-axis

\[
M = \begin{bmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Rotations in 3D

- About z-axis

\[
M = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Commuting transformations

- If A and B are matrices, does AB = BA? Always? Ever?
- What if A and B are restricted to particular transformations?
- What about the 2D transformations that we have studied?
- How about if A and B are restricted to be one of the three specific 3D rotations just introduced, such as rotation about the Z axis?

**Answer:** In general AB ≠ BA (matrix multiplication is not commutative). But if A and B are either translations or scalings, then multiplication is commutative. The same applies to rotations restricted to be about one of the 3 axis in 3D.

Rotations in 3D

- About X axis
  \[
  \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & \cos \theta & -\sin \theta & 0 \\
  0 & \sin \theta & \cos \theta & 0 \\
  0 & 0 & 0 & 1
  \end{pmatrix}
  \]

- 90 degrees about X axis
  \[
  \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1
  \end{pmatrix}
  \]

- About Y axis
  \[
  \begin{pmatrix}
  \cos \theta & 0 & \sin \theta & 0 \\
  0 & 1 & 0 & 0 \\
  -\sin \theta & 0 & \cos \theta & 0 \\
  0 & 0 & 0 & 1
  \end{pmatrix}
  \]

- 90 degrees about Y axis
  \[
  \begin{pmatrix}
  0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
  -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1
  \end{pmatrix}
  \]

- 90 degrees about X then Y
  \[
  \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1
  \end{pmatrix}
  =
  \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1
  \end{pmatrix}
  \]

- 90 degrees about Y then X
  \[
  \begin{pmatrix}
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -1 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0
  \end{pmatrix}
  =
  \begin{pmatrix}
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -1 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0
  \end{pmatrix}
  \]
Rotation about an arbitrary axis

Strategy--rotate A to Z axis, rotate about Z axis, rotate Z back to A.

As A rotates into the xz plane, its projection (shadow) onto the YZ plane (red line) rotates through the same angle which is easily calculated.

Tricky part:

1) Rotate about X to xz plane
2) Rotate about Y to Z axis.

No need to compute angles, just put sines and cosines into rotation matrices
Rotation about an arbitrary axis

Apply $R_i(\theta_i)$ to $A$ to get $A'$

$R_i(\theta_i)$ should be easy, but note that it is clockwise.

Final form is

$$R_x(-\theta_x)R_y(-\theta_y)R_z(\theta_z)R_y(\theta_y)R_x(\theta_x)$$