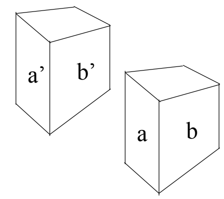
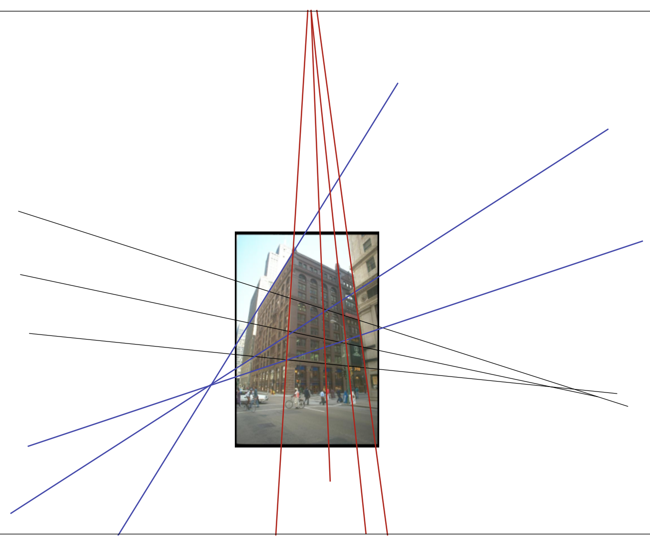
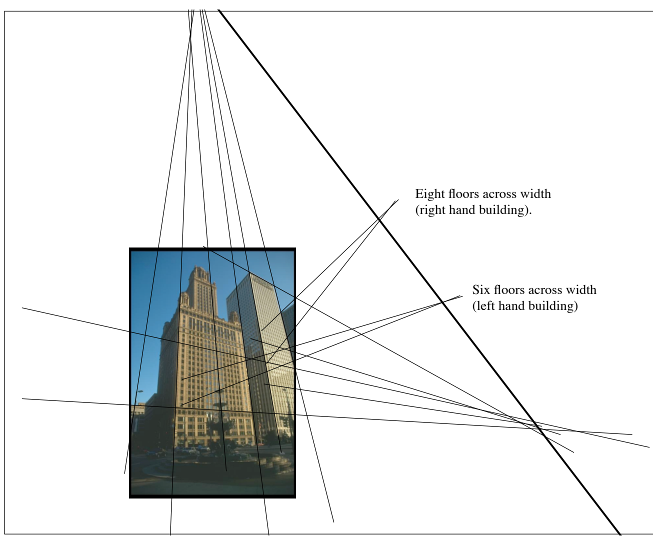
# Vanishing points

- Each set of parallel lines (=direction) meets at a different point
  - The *vanishing point* for this direction
- Sets of parallel lines on the same plane lead to *collinear* vanishing points.
  - The line is called the *horizon* for that plane
  - Standard horizon is the horizon of the ground plane.
- One way to spot fake images
  - scale and perspective don't work
  - vanishing points behave badly

Example: The figure below is claimed to provide a perspective view of two identical cubes, with faces a and a', and faces b and b' being parallel. Provide reasons why this could not be a real perspective drawing of the geometry described, marking any needed explanatory lines on the figure.





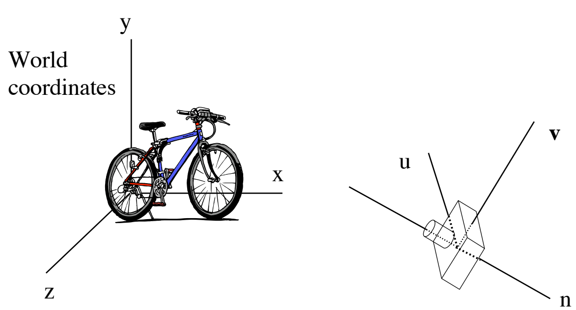
# Geometric properties of projection

- Points go to points
- Lines go to lines
- Polygons go to polygons
- Degenerate cases
  - line through focal point projects to a point
  - plane through focal point projects to a line

# Geometric Camera Model

- Let **P**=(X,Y,Z) be a point in space.
- Let (u,v) be image coordinates.
- A geometric camera model, G, tells us where P goes in the image.
- (u,v) = G(**P**)

# World and camera coordinates




# Geometric Camera Model

- Transform world coordinates to standard camera coordinates
  - (Extrinsic parameters)
- Project onto standard camera plane
  - (3D becomes 2D)
- Transform into pixel locations
  - (Intrinsic camera parameters)

Math aside, #2

## Representing Transformations

- Need mathematical representation for mapping points from the world to an image (and later, from an image taken by one camera to another).
- Represent linear transformations by matrices
- To transform a point, represented by a vector, multiply the vector by the appropriate matrix.
- To transform line segments, transform endpoints
- To transform polygons, transform vertices

## 2D Transformations

- Represent **linear** transformations by matrices
- To transform a point, represented by a vector, multiply the vector by the appropriate matrix.
- Recall the definition of matrix times vector:

$$\begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

## Matrix multiplication is linear

- In particular, if we define f(**x**)=M • **x**, where M is a matrix and **x** is a vector, then

$$\begin{aligned} f(a\mathbf{x}+b\mathbf{y}) &= M(a\mathbf{x}+b\mathbf{y}) \\ &= aM\mathbf{x}+bM\mathbf{y} \\ &= af(\mathbf{x})+bf(\mathbf{y}) \end{aligned}$$

- Where the middle step can be verified using algebra (supplementary slide)

Supplemental material

## Proof that matrix multiplication is linear

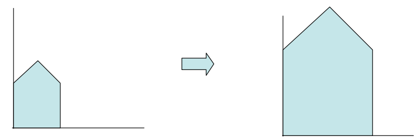
$$\begin{aligned} M(a\mathbf{x}+b\mathbf{y}) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} ax_1+by_1 \\ ax_2+by_2 \end{pmatrix} \\ &= \begin{pmatrix} a_{11}ax_1 + a_{11}by_1 + a_{12}ax_2 + a_{12}by_2 \\ a_{21}ax_1 + a_{21}by_1 + a_{22}ax_2 + a_{22}by_2 \end{pmatrix} \\ &= \begin{pmatrix} a_{11}ax_1 + a_{12}ax_2 + a_{11}by_1 + a_{12}by_2 \\ a_{21}ax_1 + a_{22}ax_2 + a_{21}by_1 + a_{22}by_2 \end{pmatrix} \\ &= a\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} + b\begin{pmatrix} a_{11}y_1 + a_{12}y_2 \\ a_{21}y_1 + a_{22}y_2 \end{pmatrix} \\ &= aM\mathbf{x}+bM\mathbf{y} \end{aligned}$$

# Composition of Transformations

- If we use one matrix, $M_1$ for one transform and another matrix, $M_2$ for a second transform, then the matrix for the first transform followed by the second transform is simply $M_2 M_1$
- This follows from the associativity of matrix multiplication
  - $M_2 (M_1\mathbf{x}) = (M_2 M_1)\mathbf{x}$
- This generalizes to any number of transforms

# Transformation examples in 2D

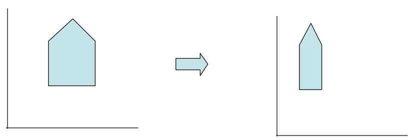
- Scale (stretch) by a factor of k



$$M = \begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix}$$ (k = 2 in the example)

# Transformation examples in 2D

- Scale by a factor of $(S_x, S_y)$



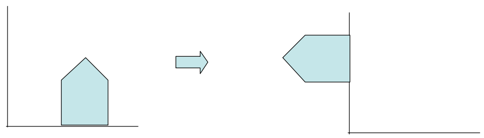
$$M = \begin{vmatrix} S_x & 0 \\ 0 & S_y \end{vmatrix}$$ (Above, $S_x = 1/2$, $S_y = 1$)

# Orthogonal Transformations

- Orthogonal transformations are defined by $O^TO=I$
- Also have $|\det(O)|=1$
- Rigid body rotations and mirror “flip”

## Transformation examples in 2D

- Rotate around origin by θ (Orthogonal)

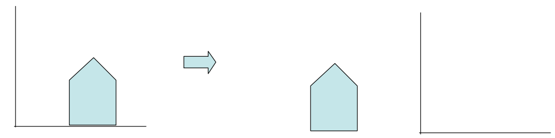


$$M = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

(Above, $\theta=90^o$)

## Transformation examples in 2D

- Mirror flip through y axis (Orthogonal)

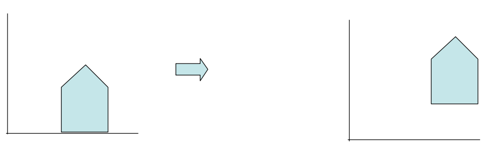


$$M = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$$

Flip over x axis is ?

## 2D Transformations

- Translation ($\mathbf{P}_{new} = \mathbf{P} + \mathbf{T}$)



M =?

## Homogenous Coordinates

- Represent 2D points by 3D vectors
- (x,y)-->(x,y,1)
- Now a multitude of 3D points (x,y,W) represent the same 2D point, (x/W, y/W, 1)
- Represent 2D transforms with 3 by 3 matrices
- Can now represent translations by matrix multiplications

## 2D Scale in H.C.

$$M = \begin{vmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

## 2D Rotation in H.C.

$$M = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

## 2D Translation in H.C.

- $\mathbf{P}_{new} = \mathbf{P} + \mathbf{T}$
- $(x', y') = (x, y) + (t_x, t_y)$

$$M = \begin{vmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{vmatrix}$$

## Transformations in 3D

- Homogeneous coordinates now have four components (x, y, z, w)
  - ordinary to homogeneous: (x, y, z) -> (x, y, z, 1)
  - homogeneous to ordinary: (x, y, z, w) -> (x/w, y/w, z/w)
- Again, translation can be expressed as a multiplication.

# Transformation examples in 3D

- Translation:

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- Anisotropic scaling:

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

# Transformation examples in 3D

- Rotation about x-axis:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

- See supplementary material for rotation about an arbitrary axis.
- A rotation matrix can be thought of as either a rotation about an axis, or a rigid transformation represented by an orthogonal matrix

# First step of geometric camera model

- Rewrite world coordinates as camera centric coordinates
  - Note that the origins are not the same and the axis are not aligned
  - Note that our rotation matrices are about an axis.
  - Hence we need to translate the world coordinates, and then rotate them.