Geometry of Single Axis Motions Using Conic Fitting

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Tutorial presentation by Kyle Simek

Motivation

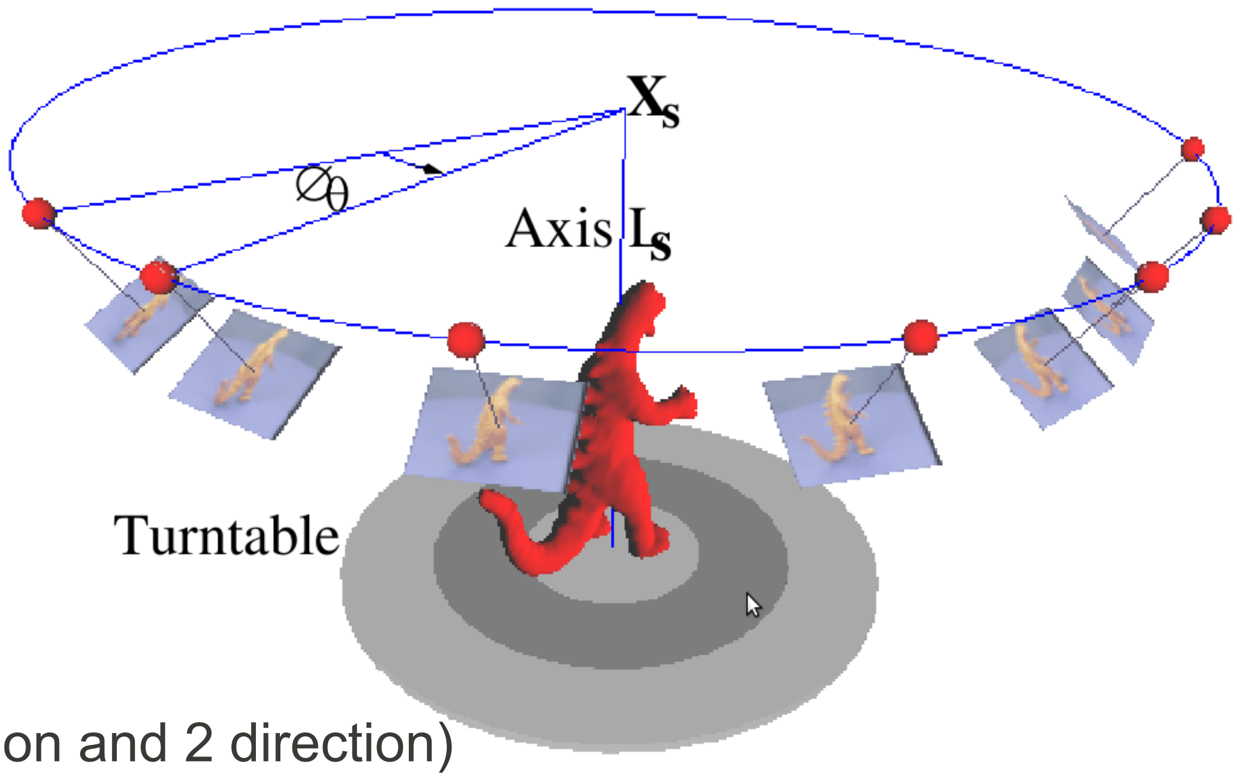
- Camera calibrations tells us where 3D points appear in an image.
- Calibrated cameras give us more information about a scene.
- Algorithms requiring calibration
  - Stereo reconstruction
  - Visual Hulls
  - Augmented Reality
- Problem: Manual calibration is tedious
  - Requires at least six correspondences between 3D points and their projection in an image.
  - More points are desirable to mitigate noise.

Auto-calibration

- Image sequences are particularly tedious to calibrate.
- Goal: infer some camera calibration parameters automatically in image sequences.
  - Correct up to a similarity transformation.
    - Can't know global scale (is it a toy dinosaur or a real one?)
    - Can't know global position (is it in Gould-Simpson or Antarctica?)
  - Exploit data redundancy/locality
  - Exploit knowledge of camera motion geometry.
- We will focus on turntable sequences.

Turntable Geometry

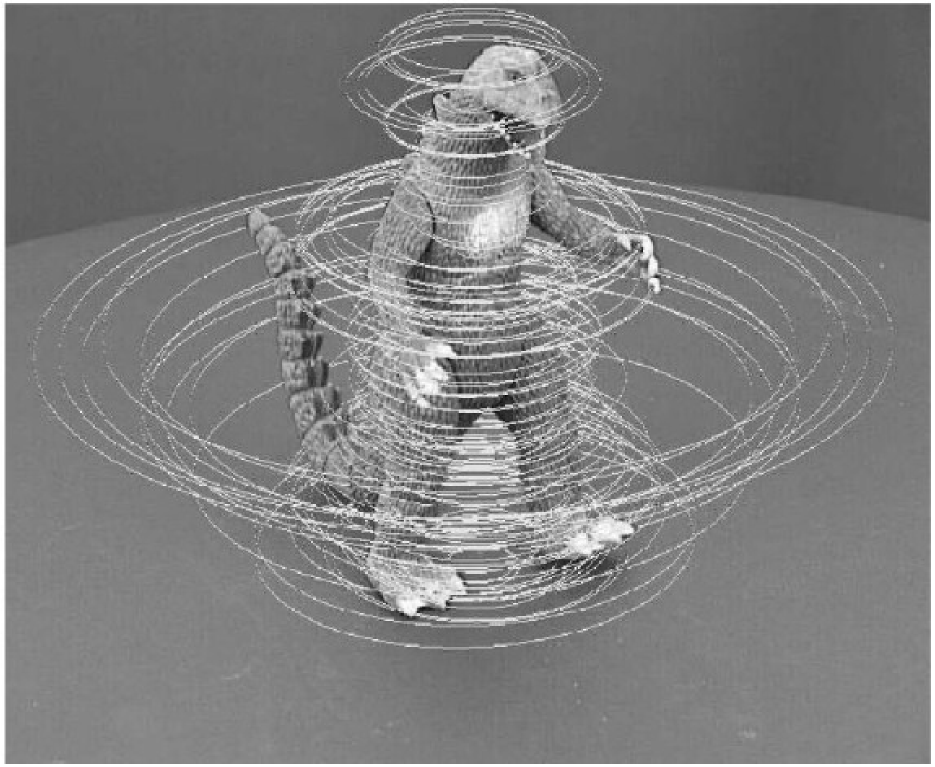
- Pure rotation around a screw axis.
  - Assume z-axis
  - Off-center from camera.
  - No translation.
- *m* angles $\theta_0 ... \theta_m$
  - Not necessarily uniform.
- Intrinsic parameters remain constant.



- Degrees of Freedom: 8 + m
  - Camera extrinsics: 4 (2 position and 2 direction)
  - Camera intrinsics: 4 (usually 5, but global scale is unknown)
  - Rotation Angles, $\theta_0 ... \theta_m$: m
  - Total: 8 + m (much better than 11m in the general case!)

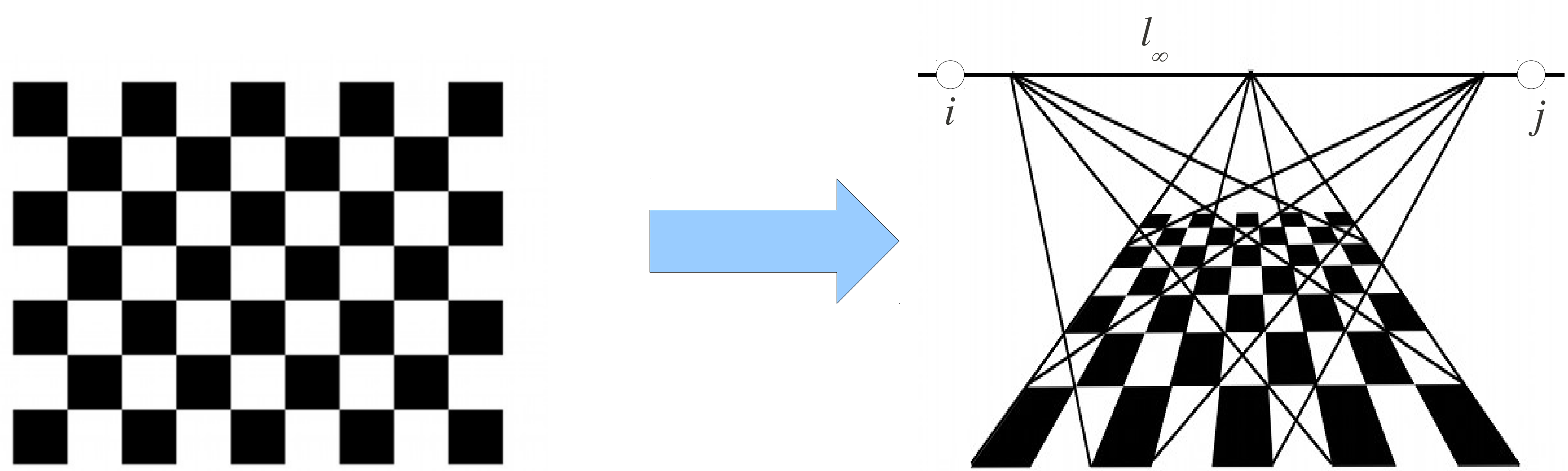
Turntable Geometry

- Object motion exhibits **unique behavior**.
  - Observation 1: All object points travel in circular paths in 3D.
    - Appear as ellipses in the image due to projection
  - Observation 2: Circular paths slice 3D space into parallel 2D horizontal planes.
- What do we know about projected circles?
- What do we know about parallel planes?



Lets review some In 2D projective geometry!

- Parallel lines intersect at the Line At Infinity, l_∞ = (0,0,1)
- All circles intersect (imaginary) circular points, $\boldsymbol{i}, \boldsymbol{j}$.
  - $\boldsymbol{i} = (1, i, 0)$; $\boldsymbol{j} = (1, -i, 0)$
  - Circular points lie on l_∞,
- Projective transform maps line at infinity and circular points into the image plane



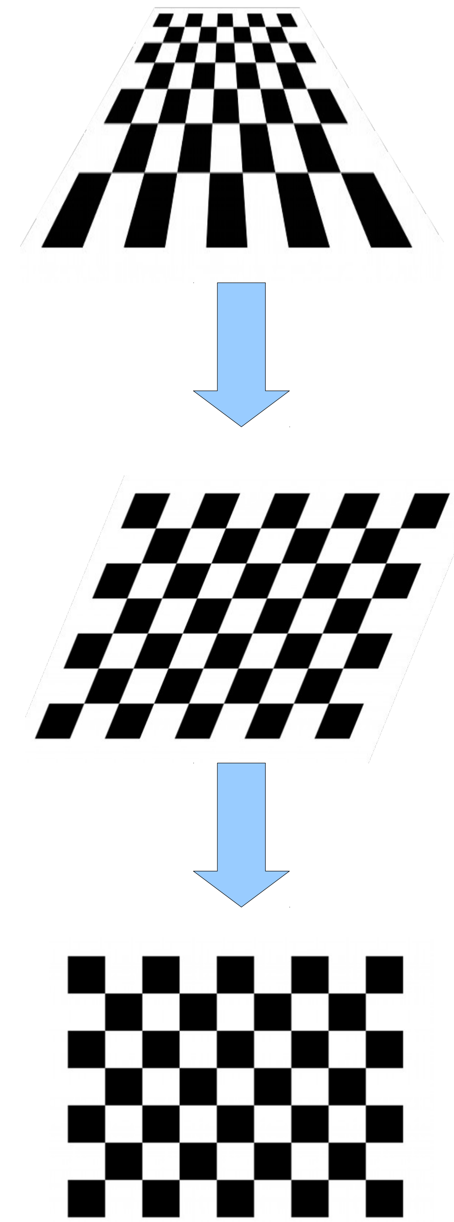
Review: Metric Reconstruction

1. Finding the line at infinity in the image, l_∞' we can map it back to l_∞ to restore affine properties (e.g. parallelism)

   1D ambiguity remains in line mapping

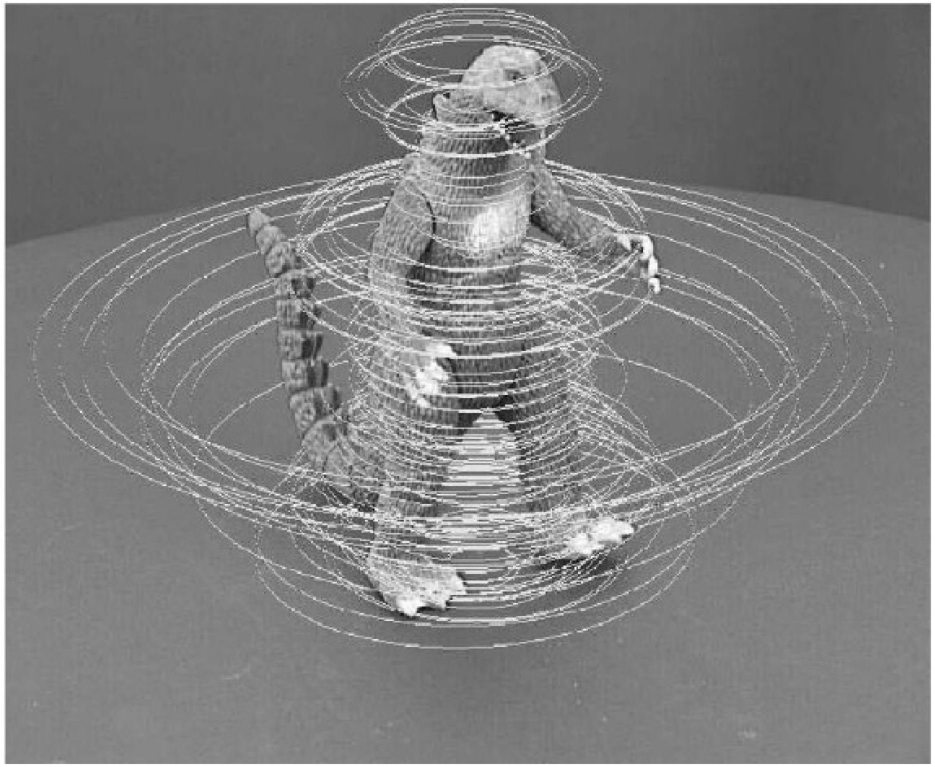
2. Mapping circular points back to canonical position restores metric properties (e.g. angles, length ratios)

   Note: Still won't know scale.



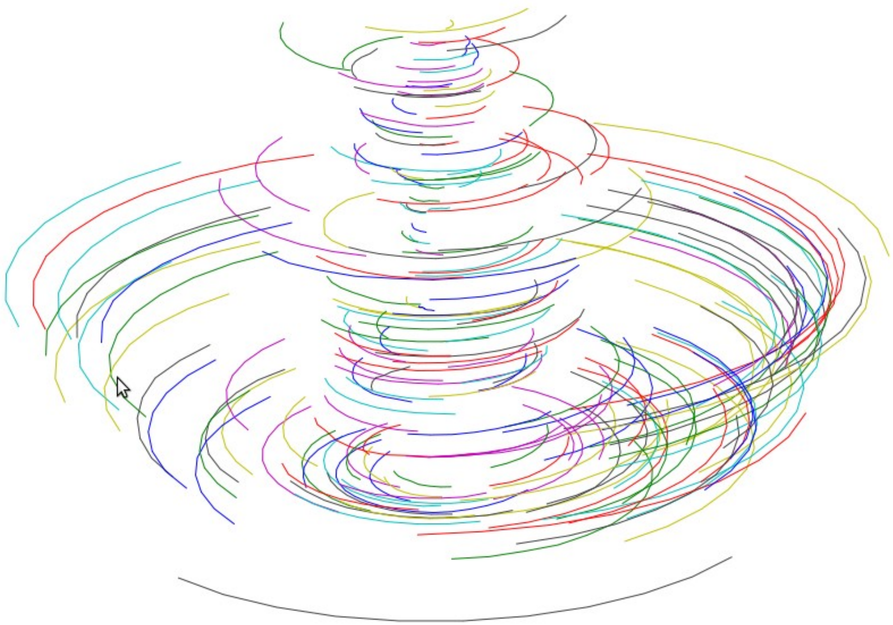
Turntable Geometry

- **Observation 1**: All object points travel in circular paths.
  - Ellipses in image.
- **Observation 2**: Circular paths slice 3D space into parallel 2D horizontal planes.
- **ALL PLANES HAVE SAME LINE-AT-INFINITY**
  - **Thus, same circular points**
- Find these features, and reconstruct the horizontal planes.



Fitting Ellipses

- Assume: we can track object points reliably.
- Goal: fit ellipses to point tracks
- Q: How many points uniquely define a conic?
  - $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
  - 5 equations, 5 unknowns
- Only need tracks of length 5
  - e.g. Find sift keypoints, find nearest neighbor within a window.
- Next: Find circular points.



Example point tracks.

Finding Circular Points in Image

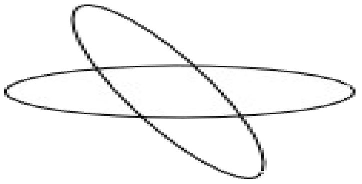
- **Key Idea 1**: Since all circles intersect the circular points, and projective transforms preserve intersections, a *projected* circle (i.e. ellipse) should intersect the *projected* circular points.
- **Key Idea 2**: Since all elliptical tracks are parallel, they all share the same circular points.
- Find the unique points where all ellipses intersect each-other.
- How can ellipses intersect?

Pop quiz!

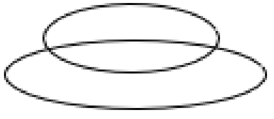
- How many times do two conics intersect?
- How many (real) intersections can two ellipses have?

Elliptical intersections

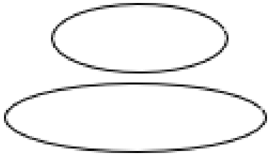
- Recall: all conics intersect at exactly four points (including duplicates, and imaginary or infinite points).
- How many (real) intersections can two ellipses have?
  - Four intersections.
    - Not possible in our configuration.



  - Two intersections
    - Could be a single intersection duplicated
    - Other two are imaginary, complex conjugates
      - **circular points!**



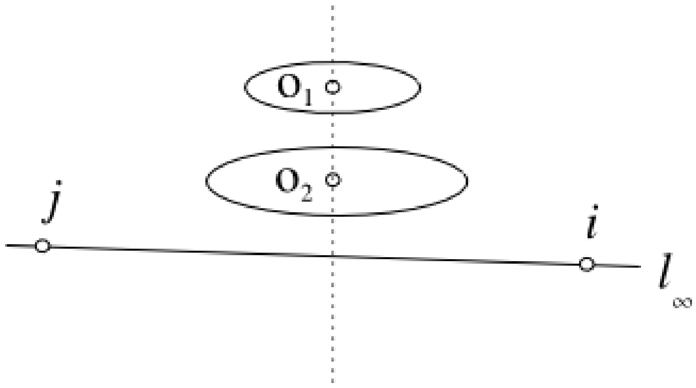
  - Zero intersection
    - Two pairs of complex conjgates
    - **Either pair could be the circular points**
    - Resolve with a third conic.



Summary So Far

- So far we have:
  1. Extracted point tracks
  2. Fit ellipses to point tracks
  3. Found circular points of the ground plane.
     - Thus, also the line at infinity
- With this, we can reconstruct the horizontal directions
  - i.e. we can make the ellipses back into circles
- Still have projective ambiguity in vertical direction.
  - i.e. We don't know the circles' relative sizes (similarity ambiguity)
- Ultimately, we still don't know:
  - Rotation angles $\theta_0 ... \theta_m$
  - Camera Matrix
- Both can be found by finding image of the circle's center.

Finding Circle's center in the Image

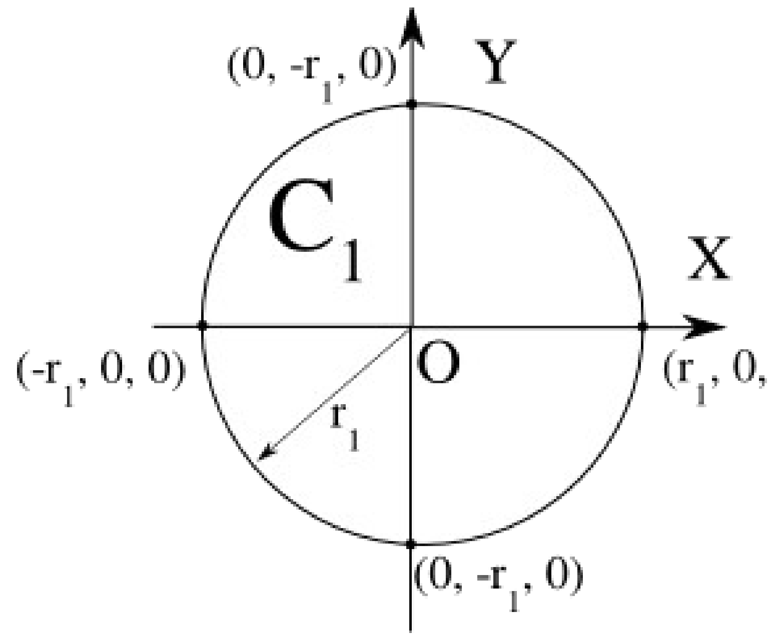


- Knowing l_∞ and an elliptical conic $\boldsymbol{C}$, we can find the circle center o.
  - (remember, $\boldsymbol{C}$ is a 3x3 matrix representing the 5-parameter conic)
- Key Idea: Poles and Polars
  - 1 to 1 mapping from a point p to a line l w.r.t. a circle $\boldsymbol{C}$.
  - Equation: $p = \boldsymbol{C}^{-1} l$
  - Demo: http://www.cut-the-knot.org/Curriculum/Geometry/PolePolar.shtml
  - **<u>Pole at circle center → polar line at infinity.</u>**
  - Poles/Polars are **<u>invariant under projection.</u>**
- The projected circle center is the pole corresponding to ellipse $\boldsymbol{C}$ and polar line l_∞.
  - Equation: $o = \boldsymbol{C}^{-1}\ l_\infty$

Finding Camera Matrix

- Recall: Camera calibration requires at least 6 3D points and corresponding 2D image points.
- We can get these points from our ellipses.
- First, we set up the *x* and *y* axis.
  - Recall that points at infinity can be interpreted as a direction vector.
  - Pick any point along l_∞. This is x_∞, the direction of the x-axis.
  - To find y_∞, we need to find vector in ground plane perpendicular to x_∞.
  - Poles and polars to the rescue! (Demo on board)
    - $L_y = C\, x_\infty$,
    - L_y intersects l_∞ at y_∞.
  - Now we have a coordinate system: x_∞, y_∞, and L_s (the turntable axis)
  - Lets pick some 3D coordinates...

3D → 2D Correspondences



(all points in inhomogeneous coordinates)

- Choose any two ellipses, C_1 and C_2.
- Set z-coordinate of C_1 be zero.
- Find intersection of lx, and ly with circle
  - These correspond to 3D points $(r_1, 0, 0)$, $(-r_1, 0, 0)$, $(0, r_1, 0)$, $(0, -r_1, 0)$
- Set z-coordinate of C_2 to be h and repeat.
  - Gives $(r_2, 0, h)$, $(-r_2, 0, h)$, $(0, r_2, h)$, $(0, r_2, h)$.
- Add point $(0,0,0)$ and $(0,0,h)$ – 10 total point correspondences.
  - Good enough to calibrate!
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Projective Ambiguity

- Result:

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & \frac{1}{h}(r_1 k_{131} + r_2 k_{132}) & r_1 k_{14} \\ p_{21} & p_{22} & \frac{1}{h}(r_1 k_{231} + r_2 k_{232}) & r_1 k_{24} \\ p_{31} & p_{32} & \frac{1}{h}(r_1 k_{331} + r_2 k_{332}) & r_1 k_{34} \end{pmatrix}$$

- Three unknowns:
  - Two circle radii: r_1 and r_2.
  - Height of circle 2: h
- One is accounted for by unknown scale.
- Other two are due to inherent projective ambiguity in Z-axis.
  - Assuming square pixels often resolves this.
- Absent any prior knowledge, of these are equally likely:

Finding Turntable Angles

- We've found the camera, but we still don't know the rotation angles $\theta_0 ... \theta_m$.
- Can be found from circle center and circular points.
- Let a, b be the image coordinates of the same object point in two consecutive time-steps.
- Laguerre's formula gives the angle between them:

  $$\theta = 1/2i \log(\{\boldsymbol{l}_{oa}, \boldsymbol{l}_{ob}; \boldsymbol{l}_{oi}, \boldsymbol{l}_{oj}\})$$

  - $\boldsymbol{l}_{oa}$, $\boldsymbol{l}_{ob}$ are line segments from the circle center (o) to a and b
  - $\boldsymbol{l}_{oi}$, $\boldsymbol{l}_{oj}$ are line segments from the circle center to the circular points
  - $\{A, B ; C, D\}$ represents the cross-ratio:
    - (A-C)(B-D) / (B – C)(A-D)
- This way, all angles can be recovered from pairs of imaged points.

Summary

1. Find tracks of 5 points
2. Fit ellipses to tracks
3. Find circular points and line at infinity
4. Find circle centers
5. Define X and Y axes
6. Extract 3D $\rightarrow$ 2D correspondences
7. Calibrate
8. Find angles using Laguerre's formula.

Extensions

- Using only two conics doesn't handle noise well.
  - Alternative: use many conics, and fit them using maximum likelihood estimation.
- Reconstruction of 3D models
  - Knowledge of cameras permits a Shape-from-silhouette reconstruction.

Questions?