Using ^ for quantities estimated from data, E() for expected value, $u(\ )$ for mean, and $\sigma^2(\ )$ for variance. By definition:

$$u(X) = E(X)$$

So estimate by:

$$\hat{u}(X) = \sum_i X_i p(x) = \sum_i X_i \left(\frac{1}{n}\right) = \frac{1}{n}\sum_i X_i \qquad \text{(equal weights)}$$

By definition

$$\sigma^2(X) = E(X - u)^2$$

Assuming equal weights, and that u is known exactly we estimate this by:

$$\hat{\sigma}^2(X) = \sum_i (X_i - u)^2 p(x) = \sum_i (X_i - u)^2 \left(\frac{1}{n}\right) = \frac{1}{n}\sum_i (X_i - u)^2$$

Normally, u is not known exactly, and you need to use $\hat{u}$ instead. Consider for a moment the variance of $\hat{u}$. First recall the formula for the variance of a linear combination of random variables (easily derived from the definition of variance as an expectation):

$$\sigma^2\left(\sum_i a_i X_i\right) = \sum_i a_i^2 \sigma^2(X_i)$$

This can be applied to the formula for $\hat{u}$ above to get

$$\sigma^2(\hat{u}) = E(u - \hat{u}) = \sum_i \sigma^2(X_i)\left(\frac{1}{n}\right)^2 = n\frac{\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Now, back to $\hat{\sigma}^2$. Recalling that variance is additive (verified by algebra using the appropriate independence assumption):

$$\sigma^2(X) = E(X - u)^2 = E(X - \hat{u})^2 + E(u - \hat{u})^2$$

Plugging in the result for the variance of the mean above:

$$\sigma^2(X) = E(X - \hat{u})^2 + \frac{\sigma^2}{n}$$

Yielding the estimate:

$$\hat{\sigma}^2(X) = \left(\frac{n}{n-1}\right) E(X - \hat{u})^2$$

And finally, we would use the estimate of $\hat{\sigma}^2(X)$ in the estimate of $\sigma^2(\hat{u})$:

$$\sigma^2(\hat{u}) = \frac{\hat{\sigma}^2(X)}{n}$$

This review has set us up for the derivations of the like quantities in the case of a weighted estimate. Note that any formula below must give the above answers if we use $w_i = \frac{1}{n}$. By definition:

$$u(X) = E(X)$$

So estimate by:

$$\hat{u}(X) = \sum_i X_i p(x) = \sum_i X_i w_i \qquad \text{(arbitrary weights)}$$

Again by definition:

$$\sigma^2(X) = E(X - u)^2$$

Assuming u is known exactly we estimate this by:

$$\hat{\sigma}^2(X) = \sum_i (X_i - u)^2 w_i = \sum_i (X_i - u)^2 w_i$$

(This is in fact what we do in the formulas in Assignment 2, to keep things simple).

To derive the analog to the correction by $\left(\frac{n}{n-1}\right)$ in the weighted case, again consider the variance of $\hat{u}$.

$$\sigma^2(\hat{u}) = E\left(\sum_i X_i w_i\right) = \sum_i \sigma^2(X_i) w_i^2 = \sigma^2(X) \sum_i w_i^2$$

As above:

$$\sigma^2(X) = E(X - u)^2 = E(X - \hat{u})^2 + E(u - \hat{u})^2$$

Plugging in the result for the variance of the mean above:

$$\sigma^2(X)\left(1 - \sum_i w_i^2\right) = E(X - \hat{u})^2$$

Rearranging:

$$\hat{\sigma}^2(X) = \frac{E(X - \hat{u})^2}{\left(1 - \sum_i w_i^2\right)} = \frac{\sum_i w_i (X_i - \hat{u})^2}{\left(1 - \sum_i w_i^2\right)}$$

Finally, we can use the estimate of $\hat{\sigma}^2(X)$ to estimate $\sigma^2(\hat{u})$:

$$\sigma^2(\hat{u}) = \hat{\sigma}^2(X) \sum_i w_i^2$$