ISTA 352

Math Tutorial 1

Administrivia

Lectures 01 and 02 posted (with demo movies!)

HW1 now posted.

TA office hours: Tuesday and Thursday 1-2 in GS 927-C

Why are we doing this?

From an information or computational science perspective, imaging and image interpretation is a mathematical topic

Several topics in the next few weeks rely on a few mathematical concepts

Currently, UA does not have an introductory level math course that covers linear algebra.

Math is good for you

Wisdom from tea dipper handle

Coordinate systems and Euclidean space

- We use coordinate systems to represent space numerically
  - The "space" could be abstract (e.g., X-axis is time, Y-axis is value of stock).
  - Today we will focus on representing concrete geometric space
- Consider a representation of where you live, and labeling where things are (e.g., a corner of a desk).
  - Notice that you used a reference point (origin)
  - Notice that order in both directions is preserved
  - The second point constrains the distortions allowed in your mapping of the world to your representation
- What else is preserved?

Coordinate systems and Euclidean space

- Our representation of space may preserve
  - Order (basically essential)
  - Angles
  - Distances
- We can represent points as vectors anchored at our origin
  - Distance between points in Euclidean space is the (Euclidean) norm of the difference between them
- Much of what we need is about changing the representations
  - Rewriting the coordinates in one space as coordinates in another
  - Computing where points in space end up in the camera sensor plane

Arrays, Vectors, and Matrices

- Arrays are N dimensional data structures where N indices extract a specific elements
- Vectors are one dimensional arrays with associated **operations**.
  - Vectors are associated with points in space relative to an origin in Euclidean space and a direction in space
- Matrices are two dimensional arrays with associated **operations**
- Vectors are also associated with Nx1 matrices (column vectors) and 1xN matrices (row vectors).

Vector and Matrix Operations

- Vector addition and subtraction
- Vector dot (inner) product
- Vector magnitude and normalization
- Vector outer product (not needed in this course)
- Vector cross product (very useful, but not needed in this course)

Vector dot product

$$\mathbf{a} \bullet \mathbf{b} = \sum a_i b_i$$

Example

$$(1\ \ 4\ \ -2) \bullet (2\ \ 2\ -1) = 1*2+4*2+(-2)*(-1)$$
$$= 2+8+2$$
$$= 12$$

Vector normalization

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}}{\sqrt{\mathbf{a} \bullet \mathbf{a}}}$$

Resulting vector has unit magnitue: $|\hat{\mathbf{a}}| = 1$

Example: Normalize (3,4,5)

Matlab output for previous slide

```
>> a=[3 4 5]
a =
    3    4    5
>> a*a'
ans =
   50
>> sqrt(a*a')
ans =
   7.0711
>> norm(a)
ans =
   7.0711
>> a_hat = a / sqrt(a*a')
a_hat =
   0.4243    0.5657    0.7071
>> a_hat * a_hat'
ans =
   1.0000
>> norm(a_hat)
ans =
    1
```

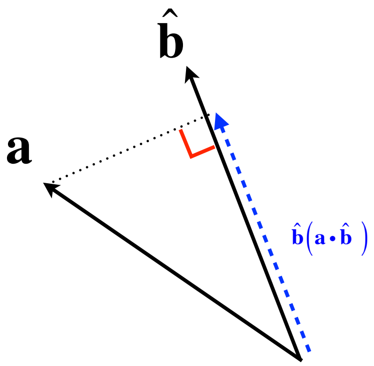
Vector dot product

The more vectors are alike (pointing in the same direction) the bigger the product.

The less information they share (i.e., the more independent) the smaller the product.

A dot product of zero means vectors are orthogonal (perpendicular).

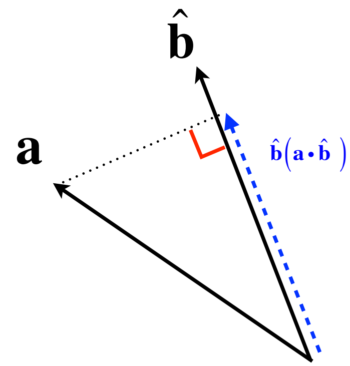
Projection



How much of **a** does **b** explain (account for)?

($\hat{\text{b}}$ is a unit vector)

Projection



Slide added after lecture

If $\hat{\mathbf{b}}$ (unit vector) is an axis of a coordinate system, then $\hat{\mathbf{b}} \cdot \mathbf{a}$ is the coordinate.
(Try it with the standard x-axis!)

Matrix Operations

- Transpose
- Matrix addition and subtraction
- Matrix-vector multiplication
- Matrix multiplication

Matrix-vector multiplication

Abstract

$$\begin{vmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \cdots\cdot \\ \mathbf{a}_n^T \end{vmatrix} * \mathbf{b} = \begin{vmatrix} \mathbf{a}_1^T \bullet \mathbf{b} \\ \mathbf{a}_2^T \bullet \mathbf{b} \\ \cdots\cdot \\ \mathbf{a}_n^T \bullet \mathbf{b} \end{vmatrix}$$

Matrix-vector multiplication

Example

$$\begin{vmatrix} 1 & 2 & 0 \\ -1 & -3 & 1 \\ 2 & 1 & -1 \end{vmatrix} * \begin{vmatrix} 2 \\ 3 \\ -1 \end{vmatrix} = ?$$

Matlab output for previous slide

```
>> A = [ 1 2 0 ; -1 -3 1 ; 2 1 -1 ]
A =
    1    2    0
   -1   -3    1
    2    1   -1

>> v=[2 3 -1]'
v =
    2
    3
   -1

>> A*v
ans =
    8
  -12
    8
```

Matrix-matrix multiplication

$$A * \begin{vmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{vmatrix} = \begin{vmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_n \end{vmatrix}$$

Or, in more detail,

$$\begin{vmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \dots. \\ \mathbf{a}_n^T \end{vmatrix} * \begin{vmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{vmatrix} = \begin{vmatrix} \mathbf{a}_1^T \bullet \mathbf{b}_1 & \mathbf{a}_1^T \bullet \mathbf{b}_2 & \dots & \mathbf{a}_1^T \bullet \mathbf{b}_n \\ \mathbf{a}_2^T \bullet \mathbf{b}_1 & \mathbf{a}_2^T \bullet \mathbf{b}_2 & \dots & \mathbf{a}_2^T \bullet \mathbf{b}_n \\ \dots. & \dots & \dots & \dots \\ \mathbf{a}_n^T \bullet \mathbf{b}_1 & \mathbf{a}_n^T \bullet \mathbf{b}_2 & \dots & \mathbf{a}_n^T \bullet \mathbf{b}_n \end{vmatrix}$$

Matrix-matrix multiplication

Example

$$\begin{vmatrix} 1 & 2 & 0 \\ -1 & -3 & 1 \\ 2 & 1 & -1 \end{vmatrix} * \begin{vmatrix} 2 & -2 & 1 \\ 3 & -1 & 0 \\ -1 & -4 & 5 \end{vmatrix} = ?$$

Matlab output for previous slide

```
>> A = [ 1 2 0 ; -1 -3 1 ; 2 1 -1 ]
A =
     1     2     0
    -1    -3     1
     2     1    -1
>> B=[2 -2 1; 3 -1 0; -1 -4 5]
B =
     2    -2     1
     3    -1     0
    -1    -4     5
>> A*B
ans =
     8    -4     1
   -12     1     4
     8    -1    -3
```

Matrix-matrix multiplication (2)

- Associative

- Not commutative!

Special matrices

- Symmetric

$$A^T = A \qquad \text{(Matlab notation A'=A)}$$

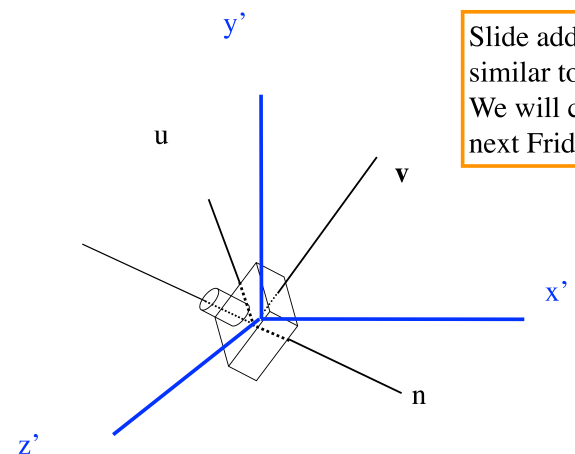
- Orthogonal
  - Columns are unit vectors orthogonal to each other

$$o_i \bullet o_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

(o_i and o_j are colums of the marix O)

$$O'*O = I$$

Bases (orthogonal)



Slide added after lecture similar to one on the board. We will come back to this next Friday.

Rewrite **x** in terms of axis in the colums of U by **x**' = U**x**