ISTA 352

Lecture 6

Tutorial on linear algebra (II)

Linear functions

- Recall notation from last lecture on camera sensors
  - **L** is light energy as a function of wavelength
  - $\mathbf{R}^{(k)}$ is energy capture sensitivity as a function of wavelength

The response is given by $\rho^{(k)} = f_{R^{(k)}}(\mathbf{L}) = \mathbf{R}^{(k)} \bullet \mathbf{L}$

The particular senstivity, $\mathbf{R}^{(k)}$, leads to the function of L, specifically $f_{R^{(k)}}(\mathbf{L})$.

In what follows, we will just call it $f(\mathbf{L})$ ($\mathbf{R}^{(k)}$ is implicit)

Linear functions

- Sensor response is linear
  - Scaling the input results in scaling the output by the same factor
    - f(a*x)=a*f(x)
  - The output of a sum of two things is the sum of the output of each individually
    - f(x+y)=f(x)+f(y)
- The usual compact formula

$$f(a*\mathbf{L_1}+b*\mathbf{L_2}) = a*f(\mathbf{L_1})+b*f(\mathbf{L}_2)$$

Linear functions

- A function imposed by a dot product is linear
  - easy to prove with algebra
- Linear functions on a vector are dot products
  - easy to prove by writing a vector as linear combination of basis vectors
- Matrix-vector multiplication
  - Our dot product function mapped vectors to numbers (one component)
  - Stacking dot products to get multiple components leads to **matrix-vector** multiplication
  - It is not hard to show that f(**v**)=M***v** is linear
    - Example, computing (R,G,B), not just R or G or B.

Linear functions (points of confusion)

- The formula is general, but we will apply it to vectors
- The formula requires a scalar multiplication operator, and an addition operator, which we inherit from vectors
- The vectors are abstract---what they represent depends on context
  - Notice that **x** and f(x) can have similar **or** different meanings
  - In this course, we will use at least three meanings
    - Spectral values (as a function of wavelength)
    - Channel intensity (R,G,B)
    - Spatial location

Linear functions (points of confusion)

- The simple transformation (*) of translating a point in space is **NOT** a linear function
  - Easy to show using the formula
  - Reasonable linear transformations satisfy f(0)=0
  - The technical term for linear followed by translation is "affine"

* Kobus will use the words "function", "transformation", and "mapping" as synonyms in this course. In addition, "linear operator" is a linear map.

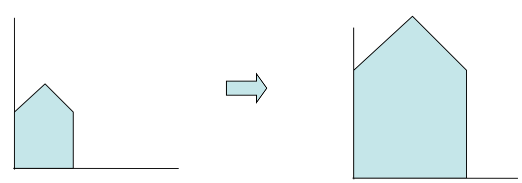
Uses of linear (and affine) operators

- Mapping points from one place to another
  - e.g., rotate a square by mapping it its corners
- Rewrite coordinates in terms of another basis (*)
  - Standard situation is our new basis is transformed version of our current basis
  - Here rewriting is the inverse of the transformation
    - If you shift a coordinate system to the right (increasing x), the new coordinates do the opposite (x decreases)

* In this course, our bases will always be orthogonal. Some statements or formulas may require orthogonality to be true.

Transformation examples in 2D

- Scale (stretch) by a factor of k

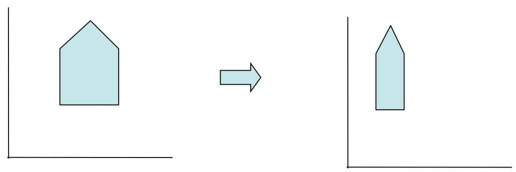


$$M = \begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix}$$

(k = 2 in the example)

Transformation examples in 2D

- Scale by a factor of (S_x, S_y)



$$M = \begin{vmatrix} S_x & 0 \\ 0 & S_y \end{vmatrix}$$

(Above, $S_x = 1/2$, $S_y = 1$)

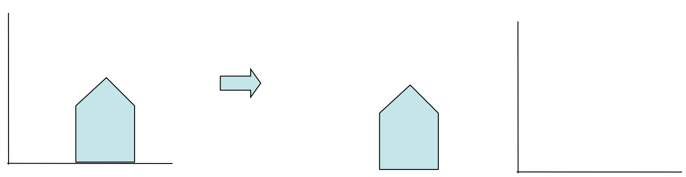
Orthogonal Transformations

- Orthogonal transformations are defined by $O^T O = I$
- Also have $|\det(O)| = 1$ (*)
- Rigid body rotations and mirror "flip"

* If you are not familiar with determinants, do not worry about it. We will not be using them in this course.

Transformation examples in 2D

- Mirror flip through y axis (Orthogonal)

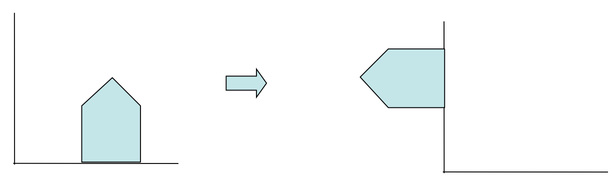


$$M = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$$

This looks like a rotation out of the page, but it is actually a bit different because which side is facing you changes in the one case but not the other.

Transformation examples in 2D

- Rotate around origin by θ (Orthogonal)



$$M = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

(Above, $\theta = 90^o$)

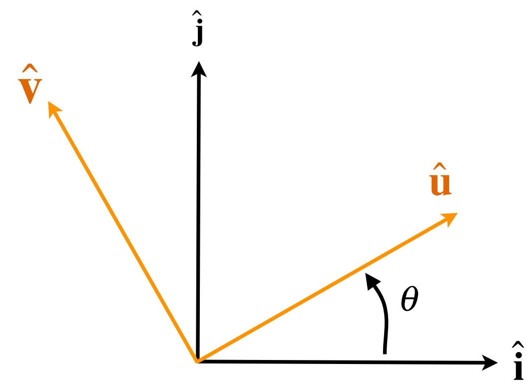
Rotating orthogonal coordinate systems

$$\left(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\right) \Rightarrow \left(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{n}}\right)$$

$$\hat{\mathbf{u}} = O\,\hat{\mathbf{i}} \qquad \hat{\mathbf{v}} = O\,\hat{\mathbf{j}} \qquad \hat{\mathbf{n}} = O\,\hat{\mathbf{k}}$$

$$O = \left|\ \hat{\mathbf{u}} \quad \hat{\mathbf{v}} \quad \hat{\mathbf{n}}\ \right|$$

Rotating orthogonal coordinate systems

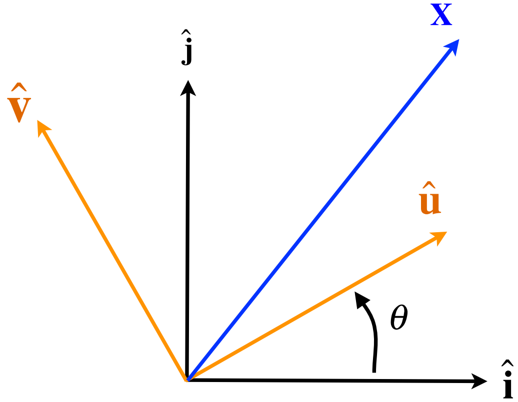


$$O = \begin{pmatrix} \hat{\mathbf{u}} & \hat{\mathbf{v}} \end{pmatrix}$$

$$\hat{\mathbf{u}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \hat{\mathbf{v}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Change of basis



$\mathbf{X} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ (in terms of first basis)

$\mathbf{X} = u\,\hat{\mathbf{u}} + v\,\hat{\mathbf{v}}$ (in terms of second)

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathbf{X}\cdot\hat{\mathbf{u}} \\ \mathbf{X}\cdot\hat{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}^T \\ \hat{\mathbf{v}}^T \end{pmatrix} \mathbf{X}$$

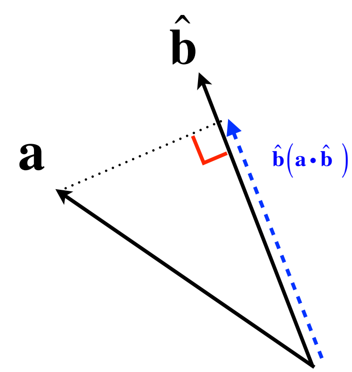
Because

$$\mathbf{X}\cdot\hat{\mathbf{u}} = \left(u\,\hat{\mathbf{u}} + v\,\hat{\mathbf{v}}\right)\cdot\hat{\mathbf{u}} = u$$

$$\mathbf{X}\cdot\hat{\mathbf{v}} = \left(u\,\hat{\mathbf{u}} + v\,\hat{\mathbf{v}}\right)\cdot\hat{\mathbf{v}} = v$$

Geometric version?

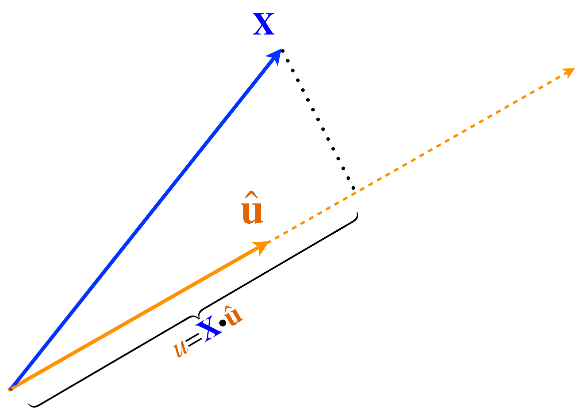
Recall Projection



If $\hat{\mathbf{b}}$ (unit vector) is an axis of a coordinate system, then $\hat{\mathbf{b}}\cdot\mathbf{a}$ is the coordinate.

(Try it with the standard x-axis!)

Coordinate with respect to a basis vector



Rotation matrix and change of basis

Rotation matrix defined by axis $(\hat{\mathbf{u}},\hat{\mathbf{v}})$ is

$$O=\begin{pmatrix} \hat{\mathbf{u}} & \hat{\mathbf{v}} \end{pmatrix}$$

Change to basis defined by axis $(\hat{\mathbf{u}},\hat{\mathbf{v}})$ is

$$O^T=\begin{pmatrix} \hat{\mathbf{u}}^T \\ \hat{\mathbf{v}}^T \end{pmatrix}$$

These are inverses, since O is orthogonal.

Pragmatic note--you usually do not need to think about angles when figuring out rotations. Just focus on where you need to go!